DEPENDENT T AND EXISTENCE OF LIMIT MODELS SH877

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Abstract. Let T be a complete first order theory and for simplicity assume G.C.H. For regular $\lambda > \kappa > |T|$ does T have (variants of) a (λ,κ)-limit models, except for stable T? For some, yes, the theory of dense linear order, for some, no. Moreover, for independent T we get negative results.

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§0 Introduction

We continue [Sh 868] and [Sh 783]
The problem in [Sh 868] is when does (first order) T have a model M of cardinality λ which is (one of the variants of) a limit model for cofinality κ, in the cases not covered by [Sh 868, 0.8] (or [Sh 88, 3.3,3.2], [Sh 88r, 3.6,3.5]. More accurately, there are some versions of limit models, "M is a $(\lambda,\kappa)-x$-limit model of T" mainly "(λ,κ)-i.m. limit", see Definition 0.7; (though we deal with others, too) the most natural case to try is $\lambda = \lambda^{<\lambda} > \kappa = \mathrm{cf}(\kappa) > |T|$.

Note that if T has (any version of) a limit model of cardinality λ then there is a universal $M \in \mathrm{Mod}_\lambda(T)$. Now we know that if $\lambda = 2^{<\lambda} > |T|$ then there is a universal $M \in \mathrm{Mod}_\lambda(T)$ (see e.g. [Ho93]). But for other cardinals it is "hard to have a universal model", see history [KjSh 409] and [Dj05]. I.e. if T has the strict order property, then there are ZFC non-existence results (a major case, for regular $\lambda(\exists\mu)(\mu^+ < \lambda \wedge 2^\mu > \lambda)$. In the main remaining cases to have a universal model, one needs special forcing).

Stable theories have limit models; hence it is natural to ask:

<u>Question 1</u>: Are there unstable T which have (λ,κ)-i.m. limit models?

In Claim 1.1 we find one: the theory T_{ord} of dense linear orders. Hence it is natural to ask:

<u>Question 2</u>: So does this hold for all unstable T?

For non-existence results it is natural to look at T dissimilar to T_{ord}.
As T_{ord} is prototypical of dependent theories, it is natural to look for independent theories. A strong, explicit version of T being independent is having the strong independence property. We prove that for such T there are no limit models (2.2). But the strong independence property does not seem a good dividing line. The independence property is a good candidate for being a meaningful dividing line.

<u>Question 3</u>: If T is independent, does T have a (λ,κ)-i.m.-limit model?

We work harder to prove (in 2.8) the negative answer for every independent T, i.e. with the independence property though a weaker version.

This makes us
<u>0.1 Conjecture</u>: Any dependent T has (λ,κ)-i.m.-limit model.

Toward this end we intend to continue the investigation of types for dependent T.

We shall also consider a property $\mathrm{Pr}_{\lambda,\kappa}(T)$ (and the stronger $\mathrm{Pr}^2_{\lambda,\kappa}(T)$), see Definition 2.4, which are relatives of "there is no $(\lambda,\kappa) - x$-limit model"; i.e. non-existence results for independent T holds (for $\lambda = \lambda^{<\lambda} \geq \kappa = \mathrm{cf}(\kappa), \lambda > |T|$. For $\lambda > \kappa$ this strengthens "there is (λ,κ)-i.m. limit model". But $\lambda = \kappa$ is a new

non-trivial case and it is also a candidate to be "an outside equivalent condition for T being dependent".

The most promising among the relatives (for having a dichotomy) is the following conjecture (the assumption $2^\lambda = \lambda^+$ is just for simplicity).
<u>0.2 The generic pair conjecture</u>: Assume $\lambda = \lambda^{<\lambda} > |T|, 2^\lambda = \lambda^+, M_\alpha \in \mathrm{EC}_T(\lambda)$ is $\prec$-increasing continuous for $\alpha < \lambda^+$ with $\cup\{M_\alpha : \alpha < \lambda^+\} \in \mathrm{EC}_T(\lambda^+)$ saturated. Then T is dependent iff for some club E of λ^+ for all pairs $\alpha < \beta < \lambda^+$ from E of cofinality $\lambda^+, (M_\beta, M_\alpha)$ has the same isomorphic type (we denote this property of T by $\mathrm{Pr}^2_{\lambda,\lambda}(T)$).

Here we prove that for independent T, a strong version of the conjecture holds. Now [Sh:F756] contains an attempt on the other case.

In §2, we also prove the parallel of what we say above. In §3 we prove that (λ, κ)-superlimit does not exist for $T = T_{\mathrm{ord}}$.

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Now we define the version of "M is a $(\lambda, S) - x$-limit model" and for it "$\bar M$ obeys a $(\lambda, T) - x$-function.

0.3 Notation. Let T denote a complete first order theory.

Let $\tau_T = \tau(T), \tau_M = \tau(M)$ be the vocabulary of T, M respectively.

0.4 Definition. 1) For any T let $\mathrm{EC}(T) = \{M : M$ a τ_T-model of $T\}$.
2) $\mathrm{EC}_\lambda(T) = \{M \in \mathrm{EC}(T) : M$ is of cardinality $\lambda\}$.
3) We say $M \in \mathrm{EC}(T)$ is λ-universal when every $N \in \mathrm{EC}_\lambda(T)$ can be elementarily embedded into M.
4) We say $M \in \mathrm{EC}(T)$ is universal when it is λ-universal for $\lambda = \|M\|$.
5) For $T \subseteq T'$ let

$$\mathrm{PC}(T', T) = \{M \restriction \tau_T : M \text{ is model of } T'\}$$

$$\mathrm{PC}_\lambda(T', T) = \{M \in \mathrm{PC}(T', T) : M \text{ is of cardinality } \lambda\}.$$

0.5 Definition. Given T and $M \in \mathrm{EC}_\lambda(T)$ we say that M is a (λ, κ)- superlimit model when: M is a λ-universal model of cardinality λ and if $\delta < \lambda^+$ a limit ordinal such that $\mathrm{cf}(\delta) = \kappa, \langle M_\alpha : \alpha \leq \delta\rangle$ is $\prec$-increasing continuous, and M_α is isomorphic to M for every $\alpha < \delta$ <u>then</u> M_δ is isomorphic to M.

0.6 Remark. We shall use:

- (a) (λ, κ)-i.md-limit in 1.1, (existence for $T_{\rm ord}$)
- (b) (λ, κ)-wk-limit in 2.2, (non-existence from "T is strongly independent")
- (c) (λ, κ)-md-limit in 2.8, (non-existence for independent T)
- (d) (λ, κ)-i.st-limit for $T_{\rm ord}$: 3.9 and 3.4(3), 3.5(3), (on characterization) for $T_{\rm ord}$)
- (e) (λ, κ)-superlimit (non-existence in 3.8).

Recall the definition of relatives of "μ limit".

0.7 Definition. Let λ be a cardinal $\geq |T|$. For parts 3) - 7) but not 8), for simplifying the presentation we assume the axiom of global choice; alternatively restrict yourself to models with universe an ordinal $\in [\lambda, \lambda^+)$. Below if $S = \{\delta < \lambda^+ : {\rm cf}(\delta) = \kappa\}$ then instead (λ, S) we write (λ, κ), this is the main case.
1) Let $S \subseteq \lambda^+$ be stationary. A model $M \in {\rm EC}_\lambda(T)$ is called (λ, S)-st-limit (or S-strongly limit or (λ, S)-strongly limit) <u>when</u> for some function: $\mathbf{F} : {\rm EC}_\lambda(T) \to {\rm EC}_\lambda(T)$ we have:

- (a) for $N \in {\rm EC}_\lambda(T)$ we have $N \prec \mathbf{F}(N)$
- (b) if $\delta \in S$ is a limit ordinal and $\langle M_i : i < \delta \rangle$ is a $\prec$-increasing continuous sequence[1] in ${\rm EC}_\lambda(T)$ obeying $\mathbf{F}$ which means $i < \delta \Rightarrow \mathbf{F}(M_{i+1}) \prec M_{i+2}$, <u>then</u> $M \cong \cup\{M_i : i < \delta\}$.

2) Let $S \subseteq \lambda^+$ be stationary. $M \in {\rm EC}_\lambda(T)$ is called (λ, S)-limit <u>if</u> for some function $\mathbf{F} : {\rm EC}_\lambda(T) \to {\rm EC}_\lambda(T)$ we have:

- (a) for every $N \in {\rm EC}_\lambda(T)$ we have $N \prec \mathbf{F}(N)$
- (b) if $\langle M_i : i < \lambda^+ \rangle$ is a $\prec$-increasing continuous sequence of members of ${\rm EC}_\lambda(T), \mathbf{F}(M_{i+1}) \prec M_{i+2}$ <u>then</u> for some closed unbounded[2] subset C of λ^+,

$$[\delta \in S \cap C \Rightarrow M_\delta \cong M].$$

$M \in {\rm EC}_\lambda(T)$ is (λ, S)-limit$^+$ <u>when</u> if $\langle M_\alpha : \alpha < \lambda^+ \rangle$ is $\subseteq$-increasing and continuous and $M_{\alpha+1} \cong M$ then for some club E of λ we have $\alpha \in E \cap S \to M_\alpha \cong M$. Notice that being a (λ, S)-limit$^+$ implies being a (λ, S)-limit.

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[1] no loss if we add $M_{i+1} \cong M$, so this simplifies the demand on $\mathbf{F}$, i.e., only $\mathbf{F}(M)$ is required
[2] we can use a filter as a parameter

3) We define "M is (λ, S)-wk-limit", "(λ, S)-md-limit" like "(λ, S)-limit", "(λ, S)-st-limit" respectively by demanding that the domain of $\mathbf{F}$ is the family of $\prec$-increasing continuous sequence of members of $\mathrm{EC}_\lambda(T)$ of length $< \lambda^+$ and replacing "$\mathbf{F}(M_{i+1}) \prec M_{i+2}$" by "$M_{i+1} \prec \mathbf{F}(\langle M_j : j \le i+1\rangle) \prec M_{i+2}$". We replace "limit" by "limit$^-$" if "$\mathbf{F}(M_{i+1}) \prec M_{i+2}$", "$M_{i+1} \prec \mathbf{F}(\langle M_j : j \le i+1\rangle) \prec M_{i+2}$" are replaced by "$\mathbf{F}(M_i) \prec M_{i+1}$", "$M_i \prec \mathbf{F}(\langle M_j : j \le i\rangle) \prec M_{i+1}$" respectively. (They are also called S-weakly limit, S-medium limit, respectively.)
4) If $S = \lambda^+$ then we may omit S (in parts (3), (4), (5)).
5) For $\Theta \subseteq \{\mu : \mu \le \lambda \text{ and } \mu \text{ is regular}\}, M$ is (λ, Θ)-strongly limit if M is $\{\delta < \lambda^+ : \mathrm{cf}(\delta) \in \Theta\}$-strongly limit in the sense of 1). Similarly for the other notions (where $\Theta \subseteq \{\mu : \mu \text{ regular } \le \lambda\}$ is non-empty and $S_1 \subseteq \{\delta < \lambda^+ : \mathrm{cf}(\delta) \in \Theta\}$ is a stationary subset of λ^+). If we do not write λ we mean $\lambda = \|M\|$.
6) We say that $M \in K_\lambda$ is (λ, S)-i.-st-limit (or S-invariantly strong limit) when in part (3), $\mathbf{F}$ is just a subset of $\{(M, N)/\cong : M \prec N \text{ are from } \mathrm{EC}_\lambda(T)\}$ and in clause (b) of part (3) we replace "$\mathbf{F}(M_{i+1}) \prec M_{i+2}$" by "$(\exists N)(M_{i+1} \prec N \prec M_{i+2} \wedge ((M, N)/\cong) \in \mathbf{F})$". But abusing notation we still write $N = \mathbf{F}(M)$ instead $((M, N)/\cong) \in \mathbf{F}$. Similarly with the other notions, i.e., we use the isomorphism type of $\bar{M}\hat{\ }\langle N\rangle$.
7) We say $\mathbf{F}$ is a (λ, T)-sequence function when...FILL! [have sequence/unary, invariant/usual]

0.8 Observation.: 1) If $\mathbf{F}_1, \mathbf{F}_2$ are as above and $\mathbf{F}_1(N) \prec \mathbf{F}_2(N)$ (or $\mathbf{F}_1(\bar{M}) \prec \mathbf{F}_2(\bar{M})$) whenever defined then if $\mathbf{F}_1$ is a witness so is $\mathbf{F}_2$.
2) All versions of limit models imply being a universal model in $\mathrm{EC}_\lambda(T)$.

0.9 Obvious implication diagram: For stationary $S \subseteq S^{\lambda^+}_\kappa$ as in 0.7(7):

$$\begin{array}{ccccc}
 & & (\lambda,\kappa)\text{-superlimit} & & \\
 & & \downarrow & & \\
 & & S\text{-strong limit} & & \\
 & \swarrow & & \searrow & \\
S\text{-medium limit,} & & & & S\text{-limit} \\
 & \searrow & & \swarrow & \\
 & & S\text{-weakly limit.} & &
\end{array}$$

0.10 Definition. T_{ord} is the theory of dense linear order with neither first nor last element.

On cuts see 3.2.

§1 Dense linear order has medium limit models

1.1 Claim. *Assume* $\lambda = \lambda^{<\lambda} > \kappa = \mathrm{cf}(\kappa)$. *Then* T_{ord} *has an invariantly medium* (λ,κ)*-limit model.*

Proof. First we say that $\bar M$ is a fast (λ,κ)-sequence (of models of T_{ord}) when:

$\circledast^\kappa_{\bar M}$ (a) $\bar M = \langle M_i : i \le \kappa\rangle$

(b) M_i is $\prec$-increasing continuous

(c) M_i is a model of T_{ord} of cardinality λ

(d) M_i is saturated if i is a non-limit

(e) if $i < \kappa$ and $a \in M_{i+1}\backslash M_i$ then $M_{i+1} \restriction \{b \in M_{i+1}\backslash M_i : (\forall c \in M_i)$ $(c < b) \equiv (c < a)\}$ is saturated

(f) if $i < \kappa$ and $A, B \subseteq M_i$ and $A < B$ (i.e. $(\forall a \in A)(\forall b \in B)(a <_{M_i} b)$ and A or B has cardinality $< \lambda$, then for some $c \in M_{i+1}\backslash M_i$ we have $A < c < B$; this includes A, B singletons but it is enough to have this when $c \in M_i \Rightarrow \neg(A < c < B)$

$(g)_1$ if $i < j < \kappa$ and $a \in M_j\backslash M_i$, then for some $d \in M_{j+1}\backslash M_j$ we have

  (α) if $b \in M_i$ and $b <_{M_j} a$ then $b <_{M_{j+1}} d$

  (β) if $c \in M_j$ and $(\forall b \in M_i)(b <_{M_j} a \Rightarrow b <_{M_j} c)$ then $d <_{M_{j+1}} c$

$(g)_2$ if $i < j < \kappa$ and $a \in M_j\backslash M_i$ then for some $d \in M_{j+1}\backslash M_j$ we have

  (α) if $b \in M_i$ and $a <_{M_j} b$ then $d <_{M_{j+1}} b$

  (β) if $c \in M_j$ and $(\forall b \in M_i)(a <_{M_j} b \Rightarrow c <_{M_j} b)$ then $c <_{M_{j+1}} d$

$(h)_1$ for $i < \kappa$ there is $b \in M_{i+1}\backslash M_i$ such that $a \in M_i \Rightarrow a <_{M_{i+1}} b$

$(h)_2$ for $i < \kappa$ there is $b \in M_{i+1}\backslash M_i$ such that $a \in M_i \Rightarrow b <_{M_{i+1}} a$

$(i)_1$ if $A \subseteq M_i, i < \kappa$ and $|A| < \lambda$ then for some $c \in M_{i+1}\backslash M_i$ we have $(\forall d \in M_i)(d <_{M_{i+1}} c \leftrightarrow (\exists a \in A)(d \le_{M_{i+1}} a)$

$(i)_2$ if $A \subseteq M_i, i < \kappa$ and $|A| < \lambda$ then for some $c \in M_{i+1}\backslash M_i$ we have $(\forall d \in M_i)(c <_{M_{i+1}} d \leftrightarrow (\forall a \in A)(a \le_{M_{i+1}} d)$.

Now clearly:

$\boxtimes$ restricting ourselves to $\mathscr{H}(\chi), \chi$ large enough, it is enough to prove $\boxtimes_1 + \boxtimes_2$ where

  $\boxtimes_1$ there is $\mathbf{F}$ such that

(α) $\mathrm{Dom}(\mathbf{F}) = \{\bar{M} : \langle M_i : i < \alpha+2\rangle$ is $\prec$-increasing, $M_i \in \mathrm{EC}_\lambda(T_{\mathrm{ord}})$ and $\alpha < \lambda^+\}$

(β) $M_{\alpha+1} \prec \mathbf{F}(\langle M_i : i < \alpha+2\rangle) \in \mathrm{EC}_\lambda(T_{\mathrm{ord}})$

(γ) $\mathbf{F}$ is invariant, i.e. if $\bar{M}_1 \cong \bar{M}_2$ then $(\bar{M}_1, \mathbf{F}(M_1)) \cong (M_2, \mathbf{F}(M_2))$

(δ) if $\bar{M} = \langle M_i : i \le \kappa\rangle$ is an $\prec$-increasing continuous sequence of members of $\mathrm{EC}_\lambda(T_{\mathrm{ord}})$ obeying $\mathbf{F}$ then $\circledast^\kappa_{\bar{M}'}$ holds for some $\bar{M}'$ satisfying $M'_\kappa = M_\kappa$

(ε) if $\delta < \lambda^+$ is a limit ordinal of cofinality κ and $\bar{M} = \langle M_i : i \le \delta\rangle$ is $\prec$-increasing continuous sequence of members of $\mathrm{EC}_\lambda(T_{\mathrm{ord}})$ obeying $\mathbf{F}$ then for some $\langle \alpha_\varepsilon : \varepsilon < \kappa\rangle$ is an increasing continuous sequence of ordinals $< \delta$ of length κ, the sequence $\bar{a}, \langle M_{\alpha_\varepsilon} : \varepsilon < \kappa\rangle\hat{\ }\langle M_\delta\rangle$ is a fast (λ,κ)-sequence

$\boxtimes_2$ if $\circledast^\kappa_{\bar{M}^1}$ and $\circledast^\kappa_{\bar{M}^2}$ then $M^1_\kappa \cong M^1_\kappa$.

Why is clause $\boxtimes_1$ true?:

How do we choose $\mathbf{F}$?

Reading the definition of $\circledast^\kappa_{\bar{M}}$ this should be clear: all our demands on M_{j+1} when we are given $\langle M_i : i \le j\rangle$ can be fulfilled. We first choose $\mathscr{P}_j = \{(A,B) : (A,B)$ a cut of M_j such that for some $c \in M_{j+1}$ we have $A < c < B\}$ satisfying clauses $(f), (g)_1, (g)_2, (h)_1, (h)_2$ of $\circledast^\kappa_{\bar{M}}$ each of them just says that $\mathscr{P}_j$ contains some set of cuts of cardinality $\le \lambda$.
Second, for each $(A,B) \in \mathscr{P}_j$ satisfies clause (e) of $\circledast^\kappa_{\bar{M}}$.
Having chosen $\mathbf{F}$, clauses $(\alpha), (\beta), (\gamma)$ of $\boxtimes_1$ follow and clause (δ) follows by clause (ε). For clause (ε) let $\langle \alpha^0_\varepsilon : \varepsilon < \kappa\rangle$ be increasing continuous with limit δ, and by induction on $\varepsilon < \kappa$ we choose α_ε by: $\alpha_\varepsilon = \cup\{\alpha_\zeta : \zeta \in \varepsilon\}$ if ε is a limit ordinal and $\min\{\alpha+1 : \alpha > \alpha_\varepsilon, \alpha > \alpha^0_\varepsilon$ and $\bar{M}$ obeys $\mathbf{F}$ at $\alpha\}$.

Why is clause $\boxtimes_2$ true?:

Suppose $\circledast^\kappa_{\langle M^\ell_i : i \le \kappa\rangle}$ for $\ell = 1,2$.

For $\ell = 1,2$ let

$$Y_\ell = \{a \in M^\ell_\kappa \backslash M^\ell_0 : \text{for every } A \subseteq M_0 \text{ of cardinality } < \lambda \text{ we have } A < a \Rightarrow (\exists b \in M^\ell_0)(A < b < a) \text{ and } a < A \Rightarrow (\exists b \in M^\ell_0)(a < b < A)\}$$

$$E_\ell = \{(a,b) : a, b \in M^\ell_\kappa \backslash M^\ell_0 \text{ and } (\forall c \in M_0)(c < a \equiv c < b)\}.$$

Now E_ℓ is an equivalence relation on $M^\ell_\kappa \backslash M^\ell_0$ and Y_ℓ is a union of some equivalence classes of E_ℓ. Let $Z_\ell \subseteq Y_\ell$ be a set of representatives of $E_\ell \restriction Y_\ell$. Now we define N_ℓ: it is the model with universe $M^\ell_0 \cup Z_\ell$, the relation $<^{N_\ell} = <^{M^\ell_\kappa} \restriction (M^\ell_0 \cup Z_\ell)$ and the relation $P^{N_\ell} = \{a : a \in M^\ell_0\}$.

Now it is easy to check that N_ℓ has first and last elements both from $N_\ell \backslash P^{N_\ell}$ and is dense. Also if $A, B \subseteq N_\ell$ have cardinality $< \lambda$ and $A < B$ then we can find a', a'' such that $A <_{N_\ell} \{a', a''\} <_{N_\ell} B$ and $a' \in P^{N_\ell}, a'' \in N_\ell \backslash P^{N_\ell}$. Hence N_ℓ is a saturated model. So easily N_1, N_2 are isomorphic and let g_0 be such an isomorphism and $f_0 = g_0 \restriction M^1_0$.

Now

- $(*)$ f_0 induces a mapping $\hat{f}_0$ from the class of E_1-equivalence classes onto the class of E_2-equivalence classes.
  [Why? Check the cases.]

Now we have to separately deal with each case of $M^1_\kappa \restriction (a_1/E_1), M^2_\kappa \restriction (a_2/E_2)$ where $\hat{f}_0(a_1/E_1) = a_2/E_2$. But this is similar to the original problem, i.e., choose $i < \kappa$ large enough such that $(a_1/E_1) \cap M^1_i \neq \emptyset$ and $(a_2/E_2) \cap M^2_i \neq \emptyset$. It is not hard to see that in the end we exhaust the models.

Let us elaborate; without loss of generality $M^1_\kappa \cap M^2_\kappa = \emptyset$. For a set $A \subseteq M^\ell_\kappa$ we define

- $(*)_1$ $E^\ell_A := \{(a,b) : a, b \in M^\ell_\kappa \backslash A \text{ and } (\forall c \in A)(a <_{M^\ell_\kappa} c \equiv b <_{M^\ell_\kappa} c)\}$.

Note

- $(*)_2$ E^ℓ_A is an equivalence relation on $M^\ell_\kappa \backslash A$.

Define

- $(*)_3$ Y^ℓ_A is the set of $\{a \in M^\ell_\kappa \backslash A$: the cut that a induces on A has cofinality (λ, λ).

So

- $(*)_4$ Y^ℓ_A is a subset of $M^\ell_\kappa \backslash A$ closed under E^ℓ_A.

Define

- $(*)_5$ We say that $A \subseteq M^\ell_\kappa$ is ℓ-nice when for every $a \in M^\ell_\kappa \backslash A$, for some $i = i_\ell(a, A) = i_\ell(a/E^\ell_A) < \kappa$ we have
  - (α) the set a/E^ℓ_A is disjoint to M^ℓ_i but not to M^ℓ_{i+1}
  - (β) the set $\{b \in A : b <_{M^\ell_\kappa} a \text{ and } b \in M^\ell_i\}$ is unbounded in

$\{b \in A : b <_{M^\ell_\kappa} a\}$

(γ) the set $\{b \in A : a <_{M^\ell_\kappa} b$ and $b \in M^\ell_i\}$ is unbounded from below in $\{b \in A : a <_{M^\ell_\kappa} b\}$

$(*)_6$ in $(*)_5$, $i_\ell(a,A)$ is uniquely defined by (a,A), actually just by a/E^ℓ_A

$(*)_7$ if $\delta < \kappa$ is a limit ordinal, $\ell \in \{1,2\}$ and $\langle A_\alpha : \alpha < \delta \rangle$ is an $\subseteq$-increasing sequence of ℓ-nice sets such that $[\alpha < \beta < \delta \wedge a \in M^\ell_\kappa \backslash A^\ell_\beta \Rightarrow i(a, A_\alpha) < i(a, A_\beta)$ then $A_\delta =: \cup\{A_\alpha : \alpha < \delta\}$ is an ℓ-nice set.

[Why? Trivially $A_\delta \subseteq M^\ell_\kappa$, so let $a \in M^\ell_\kappa \backslash A_\delta$ then for each $\alpha < \delta$ we have $a \in M^\ell_\kappa \backslash A_\alpha$ hence $i_\ell(a, A_\alpha) < \kappa$ is well defined and it is $\leq$-increasing with α. We claim $i(*) = \cup\{i_\ell(a, A_\alpha) : \alpha < \delta\}$ is as required. First of all, as the union of an increasing sequence of length $\delta < \kappa$ of ordinals $< \kappa$ it is an ordinal $< \kappa$, in fact a limit ordinal $< \kappa$.

Clearly, $a/E^\ell_{A_\delta}$ is the intersection of the decreasing sequence $\langle a/E^\ell_{A_\alpha} : \alpha < \delta \rangle$. Now if $i < i(*)$ then for some $\alpha < \delta$ we have $i \leq i_\ell(a, A_\alpha)$ hence $a/E^\ell_{A_\alpha}$ is disjoint to M^ℓ_i hence $a/E^\ell_{A_\delta} \subseteq a/E^\ell_{A_\alpha}$ is disjoint to M_i. As this holds for every $i < i(*)$ it follows that also $\cup\{M^\ell_i : i < i(*)\}$ is disjoint to $a/E^\ell_{A_\delta}$, but $\cup\{M^\ell_i : i < i(*)\} = M^\ell_{i(*)}$ because $i(*)$ is a limit ordinal. So really $(a/E^\ell_{A_\delta}) \cap M^\ell_{i(*)} = \emptyset$.

It is also clear that $(\{b \in M^\ell_{i(*)} : b <_{M^\ell_\kappa} a\}, \{b \in M^\ell_{i(*)} : a <_{M^\ell_\kappa} b\})$ is a cut of $M^\ell_{i(*)}$ whose cofinality (λ_1, λ_2) belongs to $\{(\mathrm{cf}(\delta), \mathrm{cf}(\delta)), (\lambda, \mathrm{cf}(\delta), (\mathrm{cf}(\delta), \lambda)\}$, hence by clause (f) of $\circledast^\kappa_{\bar M^\ell}$ we have $(a/E^\ell_{A_i}) \cap M_{i(*)+1} \neq \emptyset$ so $i(*)$ satisfies demand (α) from $(*)_5$ on $i(a, A_\delta)$. The other two clauses should be clear, too.]

Define

$(*)_8$ $\mathscr{F}$ is the set of f such that

(a) for some 1-nice $A_1 \subseteq M^1_\kappa$ and 2-nice set $A_2 \subseteq M^2_\kappa, f$ is an isomorphism from $M^1_\kappa \restriction A_1$ onto $M^2_\kappa \restriction A_2$

(b) for every $a_1 \in M^1_\kappa \backslash A_1$ there is $a_2 \in M^2_\kappa \backslash A_2$ such that f maps $\{b \in A_1 : b < a_1\}$ onto $\{b \in A_2 : b < a_2\}$; it follows that $a_1 \in Y^1_A$ iff $a_2 \in Y^2_A$

(c) for every $a_2 \in M^2_\kappa \backslash A_2$ for some $a_1 \in M^1_\kappa \backslash A_1$ the conclusion of clause (b) holds.

Define

$(*)_9$ $<_*$ is the following two-place relation of $\mathscr{F}$: $f <_* f'$ iff ($f, f' \in \mathscr{F}$ and)

 (a) $f \subseteq f'$

 (b) if $a_1 \in M^1_\kappa \setminus \mathrm{Dom}(f')$ then $i_1(a_1/E^1_{\mathrm{Dom}(f')}) > i_1(a_1/E^1_{\mathrm{Dom}(f)})$

 (c) if $a_2 \in M^2_\kappa \setminus \mathrm{Rang}(f')$ then $i_2(a_2/E^2_{\mathrm{Rang}(f')}) > i_2(a_2/E^2_{\mathrm{Rang}(f)}$.

Note

$(*)_{10}$ $(\mathscr{F}, <_*)$ is a non-empty partial order.
[Why? We have in $(*)$ above proved that there is an isomorphism from M^1_0 onto M^2_0 which belongs to $\mathscr{F}$.]

$(*)_{11}$ assume $f_1 <_* f_2$

 (a) if $a \in M^1_\kappa \setminus \mathrm{Dom}(f_2)$ then there are $b, c, \in (a/E^1_{\mathrm{Dom}(f_1)}) \cap \mathrm{Dom}(f_2)$ such that $b <_{M^1_\kappa} a <_{M^1_\kappa} c$

 (b) if $a \in M^2_\kappa \setminus \mathrm{Rang}(f_2)$ then there are $b, c \in (a/E^2_{\mathrm{Rang}(f_1)}) \cap \mathrm{Rang}(f_2)$ such that $b <_{M^2_\kappa} a <_{M^2_\kappa} c$

$(*)_{12}$ if $\delta < \kappa$ is a limit ordinal and $\langle f_\alpha : \alpha < \delta \rangle$ is a $<_*$-increasing sequence in $\mathscr{F}$, <u>then</u> $f_\delta := \cup\{f_\alpha : \alpha < \delta\}$ belongs to $\mathscr{F}$ and $\alpha < \delta \Rightarrow f_\alpha <_* f_\delta$.
[Why? Clearly f_δ is an isomorphism from the linear order $M^1_\kappa \restriction A^1_\delta$ where $A^1_\delta =: \cup\{\mathrm{Dom}(f_\alpha) : \alpha < \delta\}$ onto $M^2_\kappa \restriction A^2_\delta$ where $A^2_\delta =: \cup\{\mathrm{Rang}(f_\alpha) : \alpha < \delta\}$. Now $\mathrm{Dom}(f_\delta) = \cup\{\mathrm{Dom}(f_\alpha) : \alpha < \delta\}$ is 1-nice by $(*)_7$ and similarly $\mathrm{Rang}(f_\delta) = \cup\{\mathrm{Rang}(f_\alpha) : \alpha < \delta\}$ is ℓ-nice. So from the demands for "$f_\delta \in \mathscr{F}$" in $(*)_8$, clause (a) holds. Concerning clause (b) there, let $a_i \in M^1_\kappa \setminus \mathrm{Dom}(f_\delta)$. For each $\alpha < \delta$ by $(*)_{11}(a)$ applied to $f_\alpha <_* f_{\alpha+1}$ there is a pair (b_α, c_α) such that $b_\alpha, c_\alpha \in (a/E^1_{\mathrm{Dom}(f_\alpha)}) \cap \mathrm{Dom}(f_{\alpha+1}) \setminus \mathrm{Dom}(f_\alpha)$ such that $b_\alpha \subset_{M^1_\kappa} a_1 <_{M^1_\kappa} c_\alpha$. So clearly $d \in \mathrm{Dom}(f_\alpha) \Rightarrow (d < b_\alpha \equiv d < c_\alpha)$. Hence $\langle b_\alpha : \alpha < \delta \rangle$ is increasing, $\langle c_\alpha : \alpha < \delta \rangle$ is decreasing, and: $d \in \mathrm{Dom}(f_\alpha)$ implies that for every $\beta < \delta$ large enough $d < b_\alpha \equiv d < c_\alpha$. Also $b_\alpha, c_\alpha \notin \mathrm{Dom}(f_\alpha)$ so $\langle i_1(b_\alpha, \mathrm{Dom}(f_\beta)) : \beta \leq \alpha \rangle$ is increasing, $i_1(b_\alpha, \mathrm{Dom}(f_\beta)) = i_1(c_\alpha, \mathrm{Dom}(f_\beta))$. So $(\{b_\alpha; \alpha < \delta\}, \{c_\alpha : \alpha < \delta\})$ determine the cut a_1 induces on $\mathrm{Dom}(f_\delta)$ and they are $\subseteq M^1_{i(a, M^1_\kappa)}$. Now $(\{f_{\alpha+1}(b_\alpha) : \alpha < \delta\}, (\{f_{\alpha+1}(c_\alpha) : \alpha < \delta\})$, behave similarly in M^2_κ and they induce a cut of $M_i, i = \cup\{i(f_{\alpha+1}(b_\alpha), \mathrm{Rang}(f_\alpha))\}$ which is realized by some $a_2 \in M_{i+1}$ by clause (f) of $\circledast^2_{\bar{M}^2}$. Now a_2 is as required. Clause (c) is proved similarly using $(*)_{11}(b)$.]

$(*)_{13}$ if $\langle f_\alpha : \alpha < \kappa \rangle$ is an $<_*$-increasing sequence in $\mathscr{F}$ then $f_\kappa := \cup\{f_\alpha : \alpha < \kappa\}$ is an isormorphism from M^1_κ onto M^2_κ.

[Why? Toward contradiction first assume $\mathrm{Dom}(f_\kappa) \subset M^1_\kappa$ so choose $a_1 \in M^1_\kappa \backslash \mathrm{Dom}(f_\kappa)$ hence $\langle i_1(a_1/E^1_{\mathrm{Dom}(f_\alpha)}) : \alpha < \kappa \rangle$ is a (strictly) increasing sequence of ordinals $< \kappa$, hence its sup is κ, hence $i_1(a_1/E^1_{\mathrm{Dom}(f_\kappa)})$ has to be $\geq \kappa$ which is impossible. Similarly $\mathrm{Rang}(f_\kappa) \subset M^2_\kappa$ leads to contradiction, so we are done.]

$(*)_{14}$ for every $f \in \mathscr{F}$ there is f' such that $f <_* f' \in \mathscr{F}$.
[Why? Let $\langle a^1_t : t \in I_1 \rangle$ be a set of representatives of $(M^1_\kappa \backslash \mathrm{Dom}(f))$. For $t \in I$ choose $a^2_t \in M^2_\kappa \backslash \mathrm{Range}(f)$ such that f maps $\{b \in \mathrm{Dom}(f) : b < a^1_t\}$ onto $\{b \in \mathrm{Rang}(f) : b < a^2_t\}$ and let $i_{1,t} := i_1(a^1_t/E^1_{\mathrm{Dom}(f)}), i_{2,t} = i_2(a^2_t/E^2_{\mathrm{Rang}(f)})$. It is enough to choose for each $t \in I$ an isomorphism g_t from $M^1_{i_1(a^1_2,\mathrm{Dom}(f))+1} \restriction (a^1_t/E^1_{\mathrm{Dom}(f)})$ onto $M^2_{i_2(a^2_t,\mathrm{Rang}(f))+1} \restriction (a^2_t/E^2_{\mathrm{Rang}(f)})$ such that: if (A,B) is a cut of $M^1_{i_1(a^1_t,\mathrm{Dom}(f))} \restriction (a_1/E^1_{\mathrm{Dom}(f)})$ of cofinality (λ,λ) then for some $a \in M^0_1$ we have $A < a < B$ iff for some $b \in M^2_2$ we have $g_t(A) <_{M^1_2} b <_{M_2} g_t(B)$. This is done as in the proof of $(*)$ above.]

Together it follows that $M^1_\kappa \cong M^2_\kappa$ as required. $\square_{1.1}$

§2 Independent theories lack limit models

Considering §1 it is a natural to ask.

2.1 Question: 1) Is there an unstable T for which the conclusion of 1.1 fails?
2) For which unstable T does the conclusion of 1.1 fail?

Remark. 1) We shall consider also a relative $\mathrm{Pr}_{(\lambda,\kappa^*)}(\bar M)$, $\mathrm{Pr}_{\lambda,\kappa}(\bar x)$.
2) If $2^\lambda = \lambda^+$ we can restrict ourselves to $\bar M$ such that $\cup\{M_\alpha : \alpha < \lambda^+\} \in \mathrm{EC}_T(\lambda^+)$ is saturated. The union is unique (for λ) and there is $\mathbf{F}$ as in 0.7(5) guaranteeing this.

We first note that for some T's the non-existence result.

2.2 Claim. *1) If T has the strong independence property (see below, e.g. T is the theory of random graph), $|T| \le \lambda$ and $\lambda^\kappa < 2^\lambda$ then T does not have a (λ,κ)-wk-limit model.*
2) Moreover for every $\mathbf{F}$ as in Definition 0.7(5), there is a $\prec$-increasing continuous sequence $\bar M = \langle M_\alpha : \alpha < \lambda^+\rangle$ of members of $\mathrm{EC}_\lambda(T)$ obeying $\mathbf{F}$ such that if $\mathrm{cf}(\delta_1) = \kappa = \mathrm{cf}(\delta_2)$ then $M_{\delta_1} \cong M_{\delta_2} \Leftrightarrow \delta_1 = \delta_2$.

2.3 Definition. T has the strong independence property (or is strongly independent) when: for some $\varphi(\bar x, y) \in \mathbb{L}(\tau_T)$ for every $M \in \mathrm{EC}(\tau_T)$ and pairwise distinct $a_0,\dots,a_{2n-1} \in M$ for some $\bar a \in {}^{\ell g(\bar x)}M$ we have $M \models \varphi[\bar a, a_\ell]^{\mathrm{if}(\ell \text{ is even})}$.

2.4 Definition. 1) Define $S^\lambda_\kappa =: \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$. Let $\mathrm{Pr}_{\lambda,\kappa}(\bar M)$ mean that $\bar M = \langle M_\alpha : \alpha < \lambda^+\rangle$ is $\prec$-increasing continuous, each M_α is of cardinality λ and for some club E of λ^+ if $\alpha \in S^{\lambda^+}_\lambda \cap E$ and $\delta_1 \ne \delta_2 \in S^{\lambda^+}_\kappa \cap E$ but $\alpha < \delta_1 < \delta_2$ then there is no automorphism π of M_α which maps $\{\mathrm{tp}(\bar a, M_\alpha, M_{\delta_1}) : \bar a \in {}^{\omega>}(M_{\delta_1})\}$ onto $\{\mathrm{tp}(\bar a, M_\alpha, M_{\delta_2}) : \bar a \in {}^{\omega>}(M_{\delta_2})\}$ (actually even $\alpha \in E$ is O.K., i.e. we can prove it).
2) Let $\mathrm{Pr}_{\lambda,\kappa}(T)$ means: for some $\mathbf{F}$ as in 0.7(5), if $\bar M = \langle M_\alpha : \alpha < \lambda^+\rangle$ obeys $\mathbf{F}$ then $\mathrm{Pr}_{\lambda,\kappa}(\bar M)$.
3) Let $\mathrm{Pr}^2_{\lambda,\kappa}(\bar M)$ be defined as in part (1) but π is an isomorphism from M_{δ_1} onto M_{δ_2} mapping M_α onto itself. We define $\mathrm{Pr}^2_{\lambda,\kappa}(T)$ as in part (2) using $\mathrm{Pr}^2_{\lambda,\kappa}(\bar M)$. Let $\mathrm{Pr}^1_{\lambda,\kappa}(-)$ mean $\mathrm{Pr}_{\lambda,\kappa}(-)$.

2.5 Remark. 1) Clearly $\mathrm{Pr}^2_{\lambda,\kappa}(\bar M) \Rightarrow \mathrm{Pr}^1_{\lambda,\kappa}(\bar M)$ and $\mathrm{Pr}^2_{\lambda,\kappa}(T) \Rightarrow \mathrm{Pr}^1_{\lambda,\kappa}(T)$.
2) Also there is no point (in 2.4(1)) to use α_1, α_2 as some $\mathbf{F}$ guarantee that for

$\alpha_1 < \alpha_2 < \delta \in S^\lambda_\kappa$ implies there is an automorphism of M_δ mapping M_{α_1} onto M_{α_2}.

Proof. 1) Assume that $\varphi(\bar{x}, y)$ exemplifies the strong independence property.

For every $M \in \mathrm{EC}_\lambda(T)$ and (λ, T)-sequence-function $\mathbf{F}$ we can find a sequence $\langle M_\alpha : \alpha < \lambda^+\rangle$ obeying $\mathbf{F}$ such that $M \prec M_0$ and:

⊛ if $\alpha < \lambda^+$ then for some $\bar{c}^\alpha \in {}^{\ell g(\bar{x})}(M_{\alpha+1})$ we have: in $M_{\alpha+1}, a \in M_\alpha$ satisfies $\varphi(\bar{c}^\alpha, a) \Leftrightarrow a \in M$.

Now for any $\delta < \lambda^+$ of cofinality κ let $\langle \alpha^\delta_\varepsilon : \varepsilon < \kappa\rangle$ be increasing with limit δ then $\bar{\mathbf{c}}^\delta = \langle \bar{c}^{\alpha^\delta_\varepsilon} : \varepsilon < \kappa\rangle$ is a sequence of $\ell g(\bar{x})$-tuples from M_δ of length κ, and for every $a \in M_\delta$ we have:

$(*)$ a realizes the type $p(y, \bar{\mathbf{c}}^\delta) = \{\varphi(\bar{c}^{\alpha_\varepsilon}, y) : \varepsilon < \kappa\}$ in M_δ iff $a \in M$.

The number of isomorphism types of τ_T-models M of cardinality λ is 2^λ whereas the number of $\langle \bar{c}^\alpha_i : i < \kappa\rangle$ for a given M is $\le \lambda^\kappa < 2^\lambda$.

For a given $\mathbf{F}$ the construction above works for every $M \in \mathrm{EC}_\lambda(T)$, but $\dot{I}(\lambda, T) = 2^\lambda$ as $\lambda \ge |T| + \aleph_1$ so we can finish easily.
2) We can make the counterexample more explicit. For a model M and $\bar{c}^\varepsilon \in {}^{\ell g(\bar{x})}M$ for $\varepsilon < \kappa$ we define $N = N[M, \langle \bar{c}^\varepsilon : \varepsilon < \kappa\rangle]$ as the following submodel of M (if well defined) with universe the set $A = \{d \in M : M \models \varphi[\bar{c}^\varepsilon, d]$ for every $\varepsilon < \kappa\}$; (note that N is not necessarily an elementary submodel of M or even well defined, e.g. $A = \emptyset$ or A not closed under functions of M). For $M \in \mathrm{EC}_\lambda(T)$ let $\mathscr{M}[M] = \{N \prec M : N$ is $N[M, \langle \bar{c}^\varepsilon : \varepsilon < \kappa\rangle)$ for some $\bar{c}^\varepsilon \in {}^{\ell g(\bar{x})}M$ for $\varepsilon < \kappa\}$. Fixing $\mathbf{F}$ we can choose $M_\alpha \in \mathrm{EC}_\lambda(T)$ with universe $\lambda \times (1 + \alpha)$ such that

$(*)_1$ if $\alpha = 4\beta + 3$ and $\delta \le 4\beta$ then M_α is not isomorphic to $N \prec M_\delta$ whenever there are $\bar{c}^\varepsilon \in {}^{\ell g(\bar{x})}(M_\delta)$ for $\varepsilon < \kappa$ such that $N = N[M_\delta, \langle \bar{c}^\varepsilon : \varepsilon < \kappa\rangle]$

$(*)_2$ for $\alpha < \beta < \lambda^+$ there is $\bar{c}^\beta_\alpha \in {}^{(\ell g(\bar{x}))}(M_{\beta+1})$ such that for every $a \in M_\beta$ we have $M_{\beta+1} \models \varphi[\bar{c}^\beta_\alpha, a] \Leftrightarrow a \in M_\alpha$.

As $\dot{I}(\lambda, T) = 2^\lambda$ and moreover for any theory $T_1 \supseteq T$ of cardinality λ we have $\dot{I}(\lambda, T_1, T) = 2^\lambda$ and for every $M \in \mathrm{EC}_\lambda(T)$, the number of $N \in \mathscr{M}[M]$ is $\le \lambda^\kappa < 2^\lambda$ we get

⊠ for every appropriate $\mathbf{F}$ there is a $\prec$-increasing continuous sequence $\langle M_\alpha : \alpha < \lambda^+\rangle$ of models of T obeying $\mathbf{F}$ such that if $\delta_1 \ne \delta_2 < \lambda^+$ has cofinality κ <u>then</u> $M_{\delta_1}, M_{\delta_2}$ are not isomorphic.

[Why? Without loss of generality $\delta_1 < \delta_2$, let $\langle \alpha^\delta_\varepsilon : \varepsilon < \kappa \rangle$ be increasing with limit δ_2, all $> \delta_1 + 4$. Now by $(*)_2$ we know that $\langle \bar c^{\alpha_\varepsilon}_{\delta_1+3} : \varepsilon < \kappa \rangle$ exemplified that in M_{δ_2} there are $\langle \bar c^\varepsilon : \varepsilon < \kappa \rangle$ which define M_{δ_1+3}, i.e. $M_{\delta_1+3} = N[M_{\delta_2}, \langle \bar c^\varepsilon : \varepsilon < \kappa \rangle]$.

So if $M_{\delta_1} \cong M_{\delta_2}$ then there are $\bar d^\varepsilon \in {}^{\ell g(\bar x)}(M_{\delta_1})$ for $\varepsilon < \kappa$ such that $N[M_{\delta_1}, \langle \bar d^\varepsilon : \varepsilon < \kappa \rangle]$ is well defined and isomorphic to M_{δ_1+3}. But consider the choice of M_{δ_1+3}, clearly $(*)_1$ says that this is impossible.] $\square_{2.3}$

2.6 Observation. If in the definition of $\mathscr{M}[M]$ we restrict ourselves to $\langle \bar c^\varepsilon : \varepsilon < \kappa \rangle$ such that $(\forall a \in M)(\exists \varepsilon < \kappa)(\forall \zeta)(\varepsilon < \zeta < \kappa \rightarrow M \models \varphi[\bar c^\varepsilon, a] \equiv \varphi[\bar c^\zeta, a])$ then we can replace $\lambda^\kappa < 2^\lambda$ by $\lambda^{<\kappa>}$, (see [Sh 829]).

Considering 2.6 (and 1.1), it is natural to ask:
2.7 Question: Is the independence property enough to imply no limit models?

The problem was that the independence we can get may be "hidden", "camouflaged" by other "parts" of the model.
Working harder (than in 2.3), the answer is yes.

2.8 Theorem. *Assume T is independent.*
1) If $|T| \le \lambda = \lambda^\theta = 2^\kappa > \theta = \mathrm{cf}(\theta)$ then T has no (λ, θ)-i.md-limit models.
2) Moreover, there is $\mathbf{F}$ such that

- *(a) $\mathbf{F}$ is a function from $\cup\{K_\alpha : \alpha < \lambda^+ \text{ odd}\}$ where $K_\alpha = \{M : M$ a model of T with universe $\lambda \times (1+\alpha)\}$*
- *(b) if $M \in K_\alpha$ then $M \prec \mathbf{F}(M) \in K_{\alpha+1}$*
- *(c) if $M_\alpha \in K_\alpha$ for $\alpha < \lambda^+$ is $\prec$-increasing continuous and $M_{2\alpha+2} = \mathbf{F}(M_{2\alpha+1})$ then for no $\alpha < \lambda^+$ is the set $\{\delta : M_\delta \cong M_\alpha$ and $\mathrm{cf}(\delta) = \theta\}$ stationary.*

3) We can strengthen part (2) by adding in clause (c):

- *$(*)$ there are $\bar c_\alpha \in {}^\kappa(M_{2\alpha+2})$ for $\alpha < \lambda^+$ such that: if $\langle \alpha_{\ell,\varepsilon} : \varepsilon < \theta \rangle$ is an increasing continuous sequence of ordinals $< \lambda^+$ with limit α_ℓ for $\ell = 1, 2$ and $\alpha_1 \ne \alpha_2$ then there is no isomorphism f from M_{α_1} onto M_{α_2} mapping $\bar c_{\alpha_{1,\varepsilon}}$ to $\bar c_{\alpha_{2,\varepsilon}}$ for $\varepsilon < \theta$.*

4) In part (2) we can replace K_α (for $\alpha < \lambda^+$) by $K_{<\lambda^+} := \cup\{K_\alpha : \alpha < \lambda^+\}$.

2.9 Remark. 1) How does $2^\kappa = \lambda$ help us?

We shall consider $M_\alpha \in K_\alpha$ for $\alpha < \lambda^+$ which is $\prec$-increasing. We fix a sequence $\langle \bar a_t : t \in I \rangle$ in M_0 such that $\langle \varphi(x, \bar a_t) : t \in I \rangle$ is an independent set of formulas (actually $I = \lambda$). Now for any sequence $\langle \eta_i : i < \kappa + \kappa \rangle$ of members of ${}^I 2$, and

$\prec$-extension M of M_0 we can find $N, \bar{c}$ such that $M \prec N, \bar{c} = \langle c_i : i < \kappa + \kappa \rangle$ and $N \models \varphi[c_i, a_t]^{\eta_i(t)}$. Specifically if $M_{2\alpha+1}$ is already chosen then when choosing $M_{2\alpha+2}$ we choose also a sequence $\langle \eta^\alpha_i : i < \kappa + \kappa \rangle$, of members of ${}^I 2$ and $\langle c^\alpha_i : i < \kappa + \kappa \rangle$ such that $M_{2\alpha+2} \models \varphi[c^\alpha_i, a_t]^{\eta^\alpha_i(t)}$.

We may look at it as coding a sequence of λ subsets of κ. We essentially like to gain some information on $\langle \eta^\alpha_i : i < \kappa + \kappa \rangle$ from $(M_{2\alpha+1}, M_{2\alpha+2}, \bar{c}^\alpha)$, but we are not given who are the $\bar{a}_t$'s. We shall try to use $\langle c^\alpha_i : i < \kappa \rangle$, to distinguish between the "true" $\bar{a}_t$'s and "fakers". We do an approximation: some will be "exposed fakes", which we can discard, and the others are "perfect fakers", i.e., they immitate perfectly some a_t, so it does not matter.

Clearly it suffices to prove part (3) of 2.8 for having parts (1),(2) and the proof of (4) is similar. The proof is broken to some definitions and claims.

2.10 Definition. 1) Assume $\varphi = \varphi(x, \bar{y}) \in \mathbb{L}(\tau_T)$ has the independence property in T. We say $(M, \bar{\mathbf{a}})$ is a (T, φ)-candidate or an (I, T, φ)-candidate when:

- (a) M is a model of T
- (b) $\bar{\mathbf{a}} = \langle \bar{a}_t : t \in I \rangle, \bar{a}_t \in {}^{\ell g(\bar{y})} M$ and I is an infinite linear order
- (c) $\bar{\mathbf{a}}$ is an indiscernible sequence in M
- (d) $\{\varphi(x, \bar{a}_t) : t \in I\}$ is independent in M; that is for every $\eta \in \mathrm{fin}(I) := \{\eta : \eta \in {}^J 2$ for some finite $J \subseteq I\}$, there is $b \in M$ such that $t \in \mathrm{Dom}(\eta) \Rightarrow M \models \varphi[b, \bar{a}_t]^{\mathrm{if}(\eta(t))}$.

2) If $(M, \bar{\mathbf{a}})$ is an (I, T, φ)-candidate <u>then</u> let $\Gamma_{M,\bar{\mathbf{a}}} = \Gamma^{T,\varphi}_{M,\bar{\mathbf{a}}} = \Gamma^{T,\varphi,1}_{M,\bar{\mathbf{a}}} \cup \Gamma^{T,\varphi,2}_{M,\bar{\mathbf{a}}}$ be the following set of first order sentences and τ^+_M be the following vocabulary

- (a) $\tau^+_M = \tau_T \cup \{c : c \in M\} \cup \{P\}$ where P is a unary predicate ($\notin \tau_T$ of course) and each $c \in M$ serves as an individual constant ($\notin \tau_T$)
- (b) $\Gamma^{T,\varphi,1}_{M,\bar{\mathbf{a}}} = \mathrm{Th}(M, c)_{c \in M}$
- (c) $\Gamma^{T,\varphi,2}_{M,\bar{\mathbf{a}}} = \{(\exists x)[P(x) \wedge \bigwedge\limits_{t \in J} \varphi(x, \bar{a}_t)^{\eta(t)}]$ for every finite $J \subseteq I$ and $\eta \in {}^J 2\}$ (so the vocabulary is $\subseteq \tau^+_M$).

3) In (2) let $\Omega_{M,\bar{\mathbf{a}}} = \Omega^{T,\varphi}_{M,\bar{\mathbf{a}}}$ be the family of consistent sets Γ of sentences in $\mathbb{L}(\tau^+_M)$ such that Γ is of the form $\Gamma_{M,\bar{\mathbf{a}}}$ union with a subset of $\Phi_{M,\bar{\mathbf{a}}} = \{\neg(\exists x)[P(x) \wedge \psi(x, \bar{c}) \wedge \bigwedge\limits_{t \in J} \varphi(x, \bar{a}_t)^{\eta(t)}] : J \subseteq I$ is finite, $\eta \in {}^J 2, \bar{c} \in {}^{\ell g(\bar{z})} M$ and $\psi(x, \bar{z}) \in \mathbb{L}(\tau_T)\}$.

4) For $\Gamma \in \Omega_{M,\bar{\mathbf{a}}}$ let

(a) $\mathbf{S}_\Gamma = \{p : p \in \mathbf{S}(M)$ and $\Gamma \cup \{(\exists x)(P(x) \wedge \psi(x,\bar{c})) : \psi(x,\bar{c}) \in p(x)\}$ is consistent$\}$

(b) for $J \subseteq I$ and $\eta \in {}^J 2$ let $\mathbf{S}_{\Gamma,\eta} = \{p \in \mathbf{S}_\Gamma : p$ include $q^\eta_{M,\bar{\mathbf{a}}}\}$

where

(c) $q^\eta_{M,\bar{\mathbf{a}}} := q^{T,\varphi,\eta}_{M,\bar{\mathbf{a}}} = \{\varphi(x,\bar{a}_t)^{\eta(t)} : t \in \text{ Dom}(\eta)\}$.

5) For $\Gamma \in \Omega_{M,\bar{\mathbf{a}}}, \psi(x,\bar{z}) \in \mathbb{L}(\tau_T)$ and $\bar{c} \in {}^{\ell g(\bar{z})}M$ let

$$\Xi_{M,\bar{\mathbf{a}},\Gamma}(\psi(x,\bar{c})) = \{\eta \in \text{ fin}(I) : \Gamma \text{ is consistent with } (\exists x)[P(x) \wedge \psi(x,\bar{c}) \wedge \bigwedge_{t \in \text{ Dom}(\eta)} \varphi(x,\bar{a}_t)^{\eta(t)}]\}.$$

Remark. In parts (3) and (4) we could have used only $\psi(x,\bar{z}) \in \{\varphi(x,\bar{y}), \neg\varphi(x,\bar{y})\}$).

2.11 Observation. Let $(M,\bar{\mathbf{a}})$ be a (T,φ)-candidate.
1) $\Gamma_{M,\bar{\mathbf{a}}} \in \Omega_{M,\bar{\mathbf{a}}}$, i.e., $\Gamma_{M,\bar{\mathbf{a}}}$ is consistent so $\Omega_{M,\bar{a}}$ is non-empty.
2) $\Omega_{M,\bar{\mathbf{a}}}$ is closed under increasing (by $\subseteq$) unions.
3) Any member of $\Omega_{M,\bar{\mathbf{a}}}$ can be extended to a maximal member of $\Omega_{M,\bar{\mathbf{a}}}$.
4) If $M \prec N$ then $(N,\bar{\mathbf{a}})$ is a (T,φ)-candidate and for every $\Gamma \in \Omega_{M,\bar{\mathbf{a}}}$ the set $\Gamma \cup \Gamma_{N,\bar{\mathbf{a}}}$ belongs to $\Omega_{N,\bar{a}}$.
5) If $\langle I_\alpha : \alpha \leq \delta\rangle$ is an increasing continuous sequence of linear orders and $\langle N_\alpha : \alpha \leq \delta\rangle$ is $\prec$-increasing continuous sequence of models of $T, \bar{\mathbf{a}} = \langle \bar{a}_t : t \in I_\delta\rangle$ and $(N_\alpha, \bar{\mathbf{a}} \restriction I_\alpha)$ is a (T,φ)-candidate for $\alpha < \delta$ then $(N_\delta, \bar{\mathbf{a}})$ is a (T,φ)-candidate.
6) In part (5), if $\Gamma_\alpha \in \Omega_{N_\alpha,\bar{\mathbf{a}}}$ for $\alpha < \delta$ is increasing continuous with α then $\Gamma_\delta := \cup\{\Gamma_\alpha : \alpha < \delta\}$ belongs to $\Omega_{N_\delta,\bar{\mathbf{a}}}$.
7) In part (6) if Γ_α is maximal in $\Omega_{N_\alpha,\mathbf{a}}$ for each $\alpha < \delta$ then Γ_δ is maximal in $\Omega_{N_\delta,\bar{\mathbf{a}}}$.
8) If $\Gamma \in \Omega_{M,\bar{\mathbf{a}}}, \psi(x,\bar{z}) \in \mathbb{L}(\tau_T), \bar{c} \in {}^{\ell g(\bar{z})}M$ and $M \models (\exists x)\psi(x,\bar{c})$ then

(a) the empty funciton belongs to $\Xi_{M,\bar{\mathbf{a}},\Gamma}(\psi(x,\bar{c}))$

(b) if $I_1 \subseteq I_2$ are finite subsets of I and $\eta \in {}^{(I_1)}2 \cap \Xi_{M,\bar{\mathbf{a}},\Gamma}(\psi(x,\bar{c}))$ then there is $\nu \in {}^{(I_2)}2 \cap \Xi_{M,\bar{\mathbf{a}},\Gamma}(\psi(x,\bar{c}))$ extending η.

Proof. Straightforward.

2.12 Claim. *Assume that $(M, \bar{\mathbf{a}})$ is a (T, φ)-candidate and $\Gamma \in \Omega^{T,\varphi}_{M,\bar{\mathbf{a}}}$ is maximal.*
1) If $\psi(x, \bar{z}) \in \mathbb{L}(\tau_T)$ and $\bar{c} \in {}^{\ell g(\bar{z})}M$ and $\eta \in \Xi_{M,\bar{a},\Gamma}(\psi(x, \bar{c})) \subseteq \mathrm{fin}(I)$ <u>then</u> for some ν we have $\eta \subseteq \nu \in \mathrm{fin}(I)$ and $\nu \notin \Xi_{M,\bar{\mathbf{a}},\Gamma}(\neg\psi(x, \bar{c}))$.
2) For every $\eta \in {}^I2$ there are N, b such that:

- (a) *$M \prec N$ (and $\|N\| \leq \|M\| + |T|$)*
- (b) *$b \in N$*
- (c) *if $t \in I$ then $N \models \varphi[b, \bar{a}_t]^{\eta(t)}$*
- (d) *if $\bar{a} \in {}^{\ell g(\bar{z})}M, \psi = \psi(x, \bar{z}) \in \mathbb{L}(\tau_T)$ and $\psi(x, \bar{a}) \in \mathrm{tp}(b, M, N)$ <u>then</u> Γ is disjoint to $\{\neg(\exists x)[P(x) \wedge \psi(x, \bar{a}) \wedge \bigwedge_{t \in J} \varphi(x, \bar{a}_t)^{\eta(t)}] : J \subseteq I$ finite$\}$.*

Proof. 1) Assume that the conclusion fails. Consider the formula $\psi'(x, \bar{c}) := \bigwedge_{t \in \mathrm{Dom}(\eta)} \varphi(x, \bar{a}_t)^{\eta(t)} \to \neg\psi(x, \bar{c})$.

By the assumption of the claim + the assumption toward contradiction it follows that "$\rho \in \mathrm{fin}(I) \Rightarrow \Gamma \cup \{(\exists x)[P(x) \wedge \bigwedge_{t \in \mathrm{Dom}(\rho)} \varphi(x, \bar{a}_t)^{\rho(t)} \wedge \psi'(x, \bar{c})]\}$ is consistent.
[Why? Just note that it is enough to consider $\rho \in \mathrm{fin}(I)$ such that $\mathrm{Dom}(\eta) \subseteq \rho$ and we split to two cases: <u>first</u> when $\rho \restriction \mathrm{Dom}(\eta) \neq \eta$ then $\psi'(x, \bar{c})$ adds nothing in the conjunction (and use 2.10(2)(c)); <u>second</u> when $\rho \restriction \mathrm{Dom}(\eta) = \eta$ and we use the assumption toward the contradiction.]

So if N' is a model of Γ and we define N'' as N' by replacing $P^{N'}$ by $P^{N''} = \{b \in P^{N'} : N' \models \psi'[b, \bar{c}]\}$ we see that $\Gamma \cup \{\neg(\exists x)[P(x) \wedge \neg\psi'(x, \bar{c})]\} \in \Omega_{M,\bar{\mathbf{a}}}$. By the maximality of Γ it follows that $\neg(\exists x)[P(x) \wedge \neg\psi'(x, \bar{c})] \in \Gamma$. But this contradicts the assumption $\eta \in \Xi_{M,\bar{\mathbf{a}},\Gamma}(\psi(x, \bar{c}))$.
2) Easy. $\square_{2.12}$

2.13 Claim. *Assume that*

- (a) *$(M, \bar{\mathbf{a}})$ is an (I, T, φ)-candidate*
- (b) *$\bar{\eta} = \langle \eta_i : i < i(*) \rangle$ and $\eta_i \in {}^I2$ for $i < i(*)$*
- (c) *$j(*) \leq i(*)$*
- (d) *$\{\eta_i : i < j(*)\}$ is a dense subset of ${}^I2$.*

<u>Then</u> we can find $N, \bar{\mathbf{c}}$ such that

- (α) *$M \prec N$ and $\|N\| \leq \|M\| + |T| + |i(*)|$*
- (β) *$\bar{\mathbf{c}} = \langle c_i : i < i(*) \rangle$ and $c_i \in N$*

(γ) *if* $i < i(*)$ *and* $t \in I$ *then* $N \models \varphi[c_i, \bar{a}_t]^{\eta_i(t)}$

(δ) *for every* $\bar{a} \in {}^{\ell g(\bar{y})}M$ *at least one of the following holds:*

  (i) *[the perfect fakers] for some* $t \in I$ *for every* $\rho_0 \in \mathrm{fin}(I \backslash \{t\})$ *we can find* $\rho_1 \in \mathrm{fin}(I \backslash \{t\})$ *extending* ρ_0 *such that:* $\rho_1 \subseteq \eta_i \wedge i < i(*) \Rightarrow N \models$ *"*$\varphi[c_i, \bar{a}] \equiv \varphi[c_i, \bar{a}_t]$*", i.e. for "most"* $i < i(*), \bar{a}_i, \bar{a}_t$ *are similar*

  (ii) *[the rejected* $\bar{a}$*'s] for no* $t \in I$ *do we have* $i < j(*) \Rightarrow N \models \varphi[c_i, \bar{a}] \equiv \varphi[c_i, \bar{a}_t]$.

Proof. By 2.8(1), $\Gamma_{M,\mathbf{\bar{a}}} \in \Omega_{M,\mathbf{\bar{a}}}$ hence by 2.8(3) there is a maximal $\Gamma \in \Omega_{M,\mathbf{\bar{a}}}$. Let $N, \langle c_i : i < i(*) \rangle$ be such that

⊛ (a) $M \prec N$ and $\|N\| = \|M\| + |T| + |i(*)|$

  (b) for $i < i(*), c_i \in N$ realizes some $p_i \in \mathbf{S}_{\Gamma, \eta_i}$ (see Definition 2.10(4)(b)).

Clearly clauses $(\alpha), (\beta), (\gamma)$ of the desired conclusion hold, and let us check clause (δ). So assume that $\bar{a} \in {}^{\ell g(\bar{y})}M$ and clause (ii) there fails so we can choose $t \in I$ such that $i < j(*) \Rightarrow N \models$ "$\varphi[c_i, \bar{a}] \equiv \varphi[c_i, \bar{a}_t]$".

So it is enough to prove clause (i) for t; toward this assume $\rho_0 \in \mathrm{fin}(I)$ satisfies $t \notin \mathrm{Dom}(\rho_0)$, i.e. $\rho_0 \in \mathrm{fin}(I \backslash \{t\})$. Let $\rho_1 \in \mathrm{fin}(I)$ extend ρ_0 be such that $\rho_1(t) = 0$. By clause (d) of the assumption we know that for some $i < j(*)$ we have $\rho_1 \subseteq \eta_i$ but (see above) $N \models$ "$\varphi[c_i, \bar{a}] \equiv \varphi[c_i, \bar{a}_t]$" hence $\rho_1 \in \Xi_{M,\mathbf{\bar{a}},\Gamma}(\varphi(x, \bar{a})^{\eta_i(t)})$ which means that $\rho_1 \in \Xi_{M,\bar{a},\Gamma}(\varphi(x, \bar{a})^{\rho_1(t)})$. Now apply claim 2.12(1) to $\psi(x, \bar{c}) := \varphi(x, \bar{a})^{\rho_1(t)}$, so we know that for some ν we have $\rho_1 \subseteq \nu \in \mathrm{fin}(I)$ and $\nu \notin \Xi_{M,\mathbf{\bar{a}},\Gamma}(\neg\varphi(x, a)^{\rho_1(t)})$ hence

$(*)_1$ if $i < i(*)$ satisfies $\nu \subseteq \eta_i$ then $N \models \varphi[c_i, \bar{a}]^{\rho_1(t)}$ which means $N \models$ "$\varphi[c_i, \bar{a}] \equiv \varphi[c_i, \bar{a}_t]$".

Let $\rho_2 \in {}^{\mathrm{Dom}(\nu)}2$ be such that $\rho_2(t) = 1$ and $s \in \mathrm{Dom}(\nu) \backslash \{t\} \Rightarrow \rho_2(s) = \nu(s)$. We repeat the use of 2.12(1) for ρ_2 instead of ρ_1 and get ν' such that $\rho_2 \subseteq \nu' \in \mathrm{fin}(I)$ and

$(*)_2$ if $i < i(*)$ satisfies $\nu' \subseteq \eta_i$ then $N \models \neg\varphi[c_i, \bar{a}]^{\rho_2(t)}$ which means that $N \models$ "$\varphi[c_i, \bar{a}] \equiv \varphi[c_i, \bar{a}_t]$".

Let $\rho_3 = \nu' \restriction (\mathrm{Dom}(\nu') \backslash \{t\})$ and by $(*)_1 + (*)_2$ the function $\rho_3 \in \mathrm{fin}(I)$ is as required in subclause (i) (for our $\bar{a}, t, \rho_0$) in clause (δ) of the claim, so we are done. $\square_{2.13}$

2.14 Definition. 1) For a model M of $\bar{\varphi} = \varphi(x, \bar{y}) \in \mathbb{L}(\tau_M), \bar{\mathbf{c}} = \langle c_i^\varepsilon : i < i(*), \varepsilon < \theta\rangle$ such that $c_i^\varepsilon \in M$ let $\mathscr{P}_\varphi(\bar{\mathbf{c}}, M) = \{\mathscr{U} \subseteq i(*)$: for some $\bar{a} \in {}^{\ell g(\bar{y})}M$ for every $\varepsilon < \theta$ large enough $\mathscr{U} = u(\bar{a}, \langle c_i^\varepsilon : i < i(*)\rangle, M)\}$ where $u(\bar{a}, \langle c_i^\varepsilon : i < i(*)\rangle, M) = \langle i < i(*) : M \models \varphi[c_i^\varepsilon, \bar{a}]\rangle$.
2) For a model M and $\varphi = \varphi(x, \bar{y}) \in \mathbb{L}(\tau_M)$ let $\mathbf{P}^\varphi_{i(*),\theta}(M) = \{\mathscr{P}_\varphi(\bar{\mathbf{c}}, M) : \bar{\mathbf{c}} = \langle c_i^\varepsilon : i < i(*), \varepsilon < \theta\rangle$.
3) For $M_1 \prec M_2$ and $\varphi(x, \bar{y}) \in \mathbb{L}(\tau_{M_\ell})$ and $\bar{c} = \langle c_i : i < i(*)\rangle \in {}^{i(*)}M_2$ let $\mathscr{P}(\bar{\mathbf{c}}, M_1, M_2) = \{u(\bar{a}, \bar{\mathbf{c}}, M_2) : \bar{a} \in {}^{\ell g(\bar{y})}M_1\}$.

2.15 Observation. For $M, \varphi(x, \bar{y}), i(*), \theta$ as in Definition 2.14, $\mathbf{P}^\varphi_{i(*),\theta}(M)$ has cardinality $\leq \|M\|^{|i(*)|+\theta}$.

2.16 Claim. *If I is a linear order of cardinality $\leq 2^\kappa$ <u>then</u> we can find a filter D on κ and a sequence $\bar{\mathscr{U}}^* = \langle \mathscr{U}^*_{t,\ell} : t \in I\rangle, \ell \in \{0, 1, 2\}$ of members of $[\kappa]^\kappa$ such that*

- $\boxtimes_1$ *for each $t \in I, \langle \mathscr{U}^*_{t,\ell} : \ell = 0, 1, 2\rangle$ is a partition of κ and $\mathscr{U}^*_{t,2} \in D$*
- $\boxtimes_2$ *if $(M, \bar{\mathbf{a}})$ is a (I, T, φ)-candidate and $\bar{\mathscr{U}} = \langle \mathscr{U}_t : t \in I\rangle$ satisfies $\mathscr{U}_t \subseteq \mathscr{U}^*_{t,2}$ <u>then</u> for some $(N, \bar{\mathbf{c}})$ we have*
  - (a) *$M \prec N$ and $\|N\| \leq \|M\| + |\tau_T|$*
  - (b) *$\bar{\mathbf{c}} \in {}^\kappa N$*
  - (c) *if $t \in I, \bar{a} \in {}^{\ell g(\bar{y})}M$ and $\mathscr{U}^*_{t,1} \subseteq u(\bar{a}, \bar{\mathbf{c}}, N) \subset \mathscr{U}^*_{t,1} \cup \mathscr{U}^*_{t,2}$ then $u(\bar{a}, \bar{\mathbf{c}}, N) = \mathscr{U}_t$ mod D*
  - (d) *if $t \in I$ then $\mathscr{U}_t \in \mathscr{P}(\bar{\mathbf{c}}, M, N)$.*

Proof. We replace κ by $\kappa + \kappa$.

Let $\langle \eta_i^* : i < \kappa\rangle$ be a sequence of members of ${}^I\{0, 1, 2\}$ which is dense.

For $\ell = 0, 1, 2$ let $\mathscr{U}_{t,\ell} = \{i < \kappa : \eta_i(t) = \ell\}$. Notice that it is important that D is defined independently of $\mathscr{U}_{t,\ell}$ and we should therefore define it here. But for clarity of exposition we will only define it later.

Let

$$\mathscr{U}^*_{t,0} = \mathscr{U}_{t,0} \cup (\kappa + \mathscr{U}_{t,0})$$

$$\mathscr{U}^*_{t,1} = \mathscr{U}_{t,1} \cup \mathscr{U}_{t,2} \cup (\kappa + \mathscr{U}_{t,1})$$

$$\mathscr{U}^*_{t,2} = \kappa + \mathscr{U}_{t,2}.$$

Assume $\bar{\mathscr{U}} = \langle \mathscr{U}_t : t \in I \rangle$ is such that

$$\mathscr{U}^*_{t,1} \subseteq \mathscr{U}_t \subseteq \mathscr{U}^*_{t,1} \cup \mathscr{U}^*_{t,2} \subseteq \kappa + \kappa.$$

Define $\eta_i^{\bar{\mathscr{U}}} = \eta_i \in {}^{\kappa+\kappa}2$ for $i < \kappa + \kappa$

$$\eta_i(t) = \begin{cases} 0 & i \notin \mathscr{U}_t \\ 1 & i \in \mathscr{U}_t \end{cases}$$

Let $\bar{\eta} = \langle \eta_i : i < \kappa + \kappa \rangle$.
Notice that $\langle \eta_i : i < \kappa \rangle$ is dense in I_2. This is because $\mathscr{U}^*_{t,1} \subseteq \mathscr{U}_t$ and $\langle \eta^*_i : i < \kappa \rangle$ was dense in $I_{\{0,1,2\}}$. By 2.13 applied to $(M, \bar{\mathbf{a}}, \bar{\eta}), i(*) = \kappa + \kappa, j(*) = \kappa$ we can find $N, \bar{\mathbf{c}}$ as there and we should check that they are as required. Clauses $(\alpha), (\beta), (\gamma)$ of the conclusion of 2.13 give the "soft" demand.

Clearly

- $(*)$ $u(\bar{a}_t, \bar{\mathbf{c}}, N) = \{i < \kappa + \kappa : N \models \varphi[c_i, \bar{a}_t]\} = \{i < \kappa + \kappa : \eta_i(t) = 1\} = \mathscr{U}_t$ hence
- $(*)$ $t \in I \Rightarrow \mathscr{U}_t \in \mathscr{P}(\bar{\mathbf{c}}, M, N)$.

So we see that demand (d) of $\boxtimes_2$ is satisfied - all the $\mathscr{U}_t$ are included. We still need to prove (c) that is to show that there are no "fakers". So assume $\mathscr{U}^*_{t_1} \subseteq u(\bar{a}, \bar{c}, N) \subseteq \mathscr{U}^*_{t_1} \cup \mathscr{U}^*_{t_1}$ for some $t_1 \in I$ and $\bar{a} \in {}^{\ell g(\bar{y})}M$.
Denote $\mathscr{U} = u(\bar{a}, \bar{c}, N)$. We need to show $\mathscr{U} = \mathscr{U}_{t_1}$ mod D.

By clause (δ) of the conclusion of 2.13 for $\bar{a}$ one of the two clauses there (i),(ii) occurs.

Recall that

$(*)_2$ $\mathscr{U}^*_{t_1,1} \subseteq \mathscr{U} \subseteq \mathscr{U}^*_{t_1,1} \cup \mathscr{U}^*_{t_1,2}$.

So $\mathscr{U} \cap \kappa = \mathscr{U}_{t_1,1} \cap \kappa = \mathscr{U}_{t_1,1} \cup \mathscr{U}_{t_1,2}$.

Now

- $\odot$ for $\bar{a}$ clause (ii) of 2.13(δ) fails: because t_1 witnesses this by the above equality.

[Why? For each $i < \kappa$

$$i \in \mathscr{U} \Leftrightarrow i \in \mathscr{U}_{t_1,1} \cup \mathscr{U}_{t_1,2} \Leftrightarrow \eta_i[t_1] = 1 \Leftrightarrow N \models \varphi[c_i, \bar{a}_{t_1}].]$$

Conclusion: for $\bar{a}$, clause (i) of 2.13(δ) holds so there is t_2 witnessing it.
Claim

$\boxtimes_1$ $t_1 = t_2$.

Why? Toward contradiction if $t_1 \neq t_2$ then we can find $\rho_1 \in \operatorname{fin}(I \backslash \{t_2\})$ such that

$\circledast_1$ $\rho_1 \subseteq \eta_i \wedge i < \kappa + \kappa \Rightarrow N \models \varphi[c_i, \bar{a}] \equiv \varphi[c_i, \bar{a}_{t_2}]$.

Without loss of generality $t_1 \in \operatorname{Dom}(\rho_1)$ and define $\rho_2 = \rho \cup \{(t_2, 1 - \rho_1(t_1))\}$, so $\rho_1 \subseteq \rho_2 \in \operatorname{fin}(I)$. As $\{\eta^*_i : i < \kappa\}$ was chosen as a dense subset of ${}^{\{0,1,2\}}I$, there is $i < \kappa$ such that $\rho_2 \subseteq \eta^*_i$, hence by $\circledast_1$

$\circledast_2$ $N \models \varphi[c_i, \bar{a}] \equiv \varphi[c_i, \bar{a}_{t_2}]$ but

$\circledast_3$ $N \models \varphi[c_i, \bar{a}_{t_2}]^{\eta_i(t_2)}$

but $\eta_i(t_2) = 1 - \rho_1(t_1)$ hence together

$\circledast_4$ $N \models \varphi[c_i, \bar{a}]^{1-\rho_1(t_1)}$
but

$\circledast_5$ $N \models \varphi[c_i, \bar{a}_{t_1}]^{\eta_i(t_1)}$
hence

$\circledast_6$ $N \models \varphi[c_i, \bar{a}_{t_1}]^{\rho_i(t_1)}$.

But $\circledast_4 + \circledast_6$ contradict the choice of t_1 so $\boxtimes_1$ holds, i.e. $t_1 = t_2$.
Now subclause (i) of 2.13(δ) tells us

$\otimes_2$ for every $\rho_0 \in \operatorname{fin}(I \backslash \{t_1\})$ there is $\rho_1 \in \operatorname{fin}(I \backslash \{t_1\})$ extending ρ_0 such that
- (a) $\rho_1 \subseteq \eta_i \wedge i < \kappa + \kappa \Rightarrow N \models \varphi[\bar{c}_i, \bar{a}] \equiv \varphi[c_i, \bar{a}_{t_1}]$ hence
- (b) if $\rho_1 \subseteq \eta_i \wedge i < \kappa + \kappa \Rightarrow i \in u_\varphi(\bar{a}, \bar{\mathbf{c}}, N) \Leftrightarrow i \in \mathscr{U}_{t_1}$.

So let

$$\begin{aligned} D = \{\mathscr{U} \subseteq \kappa + \kappa : & \text{ for every } \rho_0 \in \operatorname{fin}(I) \text{ there is } \rho_1, \\ & \rho_0 \subseteq \rho_1 \in \operatorname{fin}(I) \text{ such that} \\ & \kappa \leq i < \kappa + \kappa \wedge \rho_1 \subseteq \eta_i \Rightarrow i \in \mathscr{U}\}. \end{aligned}$$

Clearly we are done. $\square_{2.16}$

Proof of the Theorem 2.8(3). Like the proof 2.2 of the case "T has the strong independence property."

2.17 Remark. The $\mathbf{F}$ we construct works for all $\theta = \mathrm{cf}(\theta) < \lambda$ for which $\lambda = \lambda^\theta$ simultaneously.

2.18 Discussion: Can we prove 2.8 also for λ strongly inaccessible? Toward this

- (a) we have to use $\bar{\mathbf{c}}_\alpha = \langle c_{\alpha,i} : i < \lambda \rangle$, instead $\langle c_{\alpha,i} : i < \kappa \rangle$
- (b) each M_α has a presentation $\langle M_{\alpha,\zeta} : \zeta < \lambda \rangle$
- (c) for a club E of $\mu < \lambda$, we use $\langle c_{\alpha,i} : i < \mu \rangle \hat{\ } \langle c_\mu \rangle$ to code $\mathscr{U}_\alpha \cap \mu$
- (d) instead $i, \kappa + i$ we use $2i, 2i+1$.

So the problem is: arriving to μ, we have already committed ourselves for the coding of $\mathscr{U}_\alpha \cap \mu'$ for $\mu' \in E_\alpha \cap \mu$, what freedom do we have in μ?

Essentially we have a set $\Lambda_\mu \subseteq {}^{2^\mu}2$ quite independent, and for $\mu_1 < \mu_2$, there is a natural reflection, the set of possibilities in ${}^\lambda 2$ is decreasing. But the amount of freedom left should be enough to code. We shall deal in [Sh:F792] with the inaccessible case.

Question: Can we improve as $(*)_1$ in the proof of 2.8(3) in the case of strongly dependent?

2.19 Claim. *1) Assume T has the strong independence property. If $\lambda \ge \kappa = \mathrm{cf}(\kappa), 2^{\min\{2^\kappa,\lambda\}} > \lambda^\kappa$ and $\lambda > |T|$, then $\mathrm{Pr}_{\lambda,\kappa}(T)$.*
2) Assume T is independent. If λ, κ are as above, then $\mathrm{Pr}(\lambda,\kappa)$. $\square_{3.9}$

Proof. 1) Let $\varphi(\bar{x}, y)$ exemplify "T has the strong independent property", see Definition 2.3.

We choose $\mathbf{F}$ such that:

$(*)$ if $\mathbf{F}(\langle M_i : i \le \alpha + 1 \rangle) \prec M_{\alpha+2}$ then for every $i \le \alpha$ for some $\bar{c} = \bar{c}_{\alpha,i} \in {}^{\ell g(\bar{x})}(M_{\alpha+2})$ the set $\{a \in M_i : M_{\alpha+1} \models \varphi[\bar{c}, a]\}$ does not belong to $\{\{a \in M_i : M_{\alpha+1} \models \varphi[\bar{d}, a]\} : \bar{d} \in {}^{\ell g(\bar{x})}(M_{\alpha+1})\}$.

We continue as in the proof of 2.2.
2) Similarly (recall the proof of 2.8). $\square_{2.19}$

§3 More on (λ, κ)-limit for T_{ord}

It is natural to hope that a (λ, κ)-i.md.-limit model is (λ, κ)-superlimit but in Theorem 3.8. we prove that there is no (λ, κ)-superlimit model for T_{rd}.

We conclude by showing that the (λ, κ)-i.md.-limit model has properties in the direction of superlimit. By 3.9 it is (λ, S)-limit$^+$, that is if $\langle M_\alpha : \alpha < \lambda^+ \rangle$ is a $\subseteq$-increasing sequence of (λ, κ)-i.md-limit models for a club of $\delta < \lambda^+$ of cofinality κ the model $\cup\{M_i : i < \delta\}$ is a (λ, κ)-i.md.-limit model. Also in §1 the function $\mathbf{F}$ does not need memory.

3.1 Hypothesis. 1) $\lambda = \lambda^{<\lambda} > \kappa = \mathrm{cf}(\lambda)$.
2) T_{rd} is the theory of linear order, (recall that T_{ord} the theory of dense linear order with neither first nor last element).
3) We deal with $\mathrm{EC}_{T_{\mathrm{rd}}}(\lambda)$, ordered by $\subseteq$, so M, N denotes members of $\mathrm{EC}_{T_{\mathrm{rd}}}(\lambda)$.

3.2 Definition. 1) We say that (C_1, C_2) is a cut of $M \in \mathrm{EC}_{T_{\mathrm{rd}}}$ when:

- (a) C_1 is an initial segment of M
- (b) C_2 is an end-segment of M
- (c) $C_1 \cap C_2 = \emptyset$
- (d) $C_1 \cup C_2 = M$.

2) For a cut (C_1, C_2) of M, let $\mathrm{cf}(C_1, C_2)$, the cofinality of the cut (C_1, C_2), be the pair (θ_1, θ_2) when

- (a) θ_1 is the cofinality of C_1, i.e. of $M \restriction C_1$ (can be $0, 1$ or a regular cardinal $\in [\aleph_0, \lambda)$
- (b) θ_2 is the cofinality of C_2 inverted.

3) If $M \subseteq N$ and (C_1, C_2) is a cut of M let $N^{[(C_1,C_2)]} = N \restriction \{a \in N : a$ realizes the cut (C_1, C_2) of M which means $c_1 \in C_1 \Rightarrow c_1 <_N a$ and $c_2 \in C_2 \Rightarrow a <_N c_2$.
4) For a cut (C_1, C_2) of M, A is unbounded in the cut if $A \cap C_1$ is unbounded in C_1 and $A \cap C_2$ is unbounded from below in C_2.

* * *

3.3 Definition. 1) We say $\bar{M}$ is a (λ,κ)-sequence <u>when</u>:

(a) $\bar{M} = \langle M_i : i \le \kappa\rangle$ is $\subseteq$-increasing continuous sequence of members of $\mathrm{EC}_{T_{\mathrm{rd}}}(\lambda)$

(b) if $i < \kappa$ and (C_1, C_2) is a cut of M_i then (α) or (β) hold but not both where

  (α) $\mathrm{cf}(C_1, C_2) = (\lambda,\lambda)$ and no $a \in M_\kappa \backslash M_i$ realizes (C_1, C_2)

  (β) some $a \in M_\kappa \backslash M_i$ realizes (C_1, C_2) (it follows that $M_\kappa \restriction \{a \in M_j : a$ realizes $(C_1, C_2)\}$ is infinite)

(c) for every $a <_{M_i} b$ the model $(M_\kappa \restriction (a,b)_{M_\kappa})$ is universal (for $\mathrm{EC}_{T_{\mathrm{rd}}}(\lambda)$, usual embedding).

Remark. Compared to §1 we do not require

(d) if $i < \kappa$ and (C_1, C_2) is a cut of M_i not realized by any $a \in M_\kappa$ <u>then</u>: for every $j < i$, either M_j is unbounded in (C_1, C_2), or for some $a_1 \in C_1, a_2 \in C_2$ the interval $(a_1, a_2)_{M_i}$ is disjoint to M_j.

3.4 Claim. *1) If $M = \langle M_i : i \le \kappa\rangle$ is a (λ,κ)-sequence <u>then</u> M_κ is (λ,κ)-i.md.-limit (and so for some $\bar{M}' = \langle M'_i : i \le \kappa\rangle$ such that $\circledast_{\bar{M}'}$ from the proof of 1.1 holds and $M_\kappa \cong M'_\kappa$).*
2) If (C_1, C_2) is a cut of $M_i, i < \kappa$ and $(b)(\beta)$ of Definition 3.3 holds, then for every $j \in (i+3,\kappa), M_j \restriction \{a \in M_j : a$ realizes $(C_1, C_2)\}$ is a universal model of T_{rd}.
3) If M is (λ,κ)-i.md.-limit, <u>then</u> there is a (λ,κ)-sequence $\langle M_i : i \le \kappa\rangle$ such that $M_\kappa = M$.
4) If $M \in \mathrm{EC}_{T_{\mathrm{rd}}}(\lambda)$ then:

(a) *if $\lambda = \|M\| = \lambda^{<\lambda}$ <u>then</u> the number of cuts of M of cofinality $\ne (\lambda,\lambda)$ is at most λ*

(b) *if $\lambda = \|M\| = \|M\|^\kappa$ <u>then</u> the number of cuts of M of cofinality (κ,κ) is at most λ*

(c) *if $\lambda = \|M\|$ <u>then</u> the number of cuts of cofinality (σ_1,σ_2) where $\sigma_1 \ne \sigma_2$ is $\le \lambda$.*

Proof. 1) As in the proof of 1.1, see 3.5(1) -Saharon: FILL!
2) There is $c \in M^{[c_1,c_2]}_{i+1}$ and $d \in M_{i+2}$ such that $c < d, (c,d) \cap M_{i+1} = \emptyset$ and $M_{i+3} \restriction (c,d)$ is universal.

3) Should be clear.
4) Clauses (a),(b) are easy and clause (c) holds by [Sh:c, VIII,§0].

Remark. A difference between Definition 3.3 and the earlier one is that we do not ask that a dense set of cuts of cofinality (λ, λ) of M_i is realized in $\cup\{M_j : j < i\}$.

3.5 Observation. 1) If $M \in \mathrm{EC}_{T_{\mathrm{rd}}}(\lambda)$ is univeral, $\kappa < \lambda$ and $M = \bigcup\limits_{i<\kappa} I_i$ <u>then</u> for at least one $i < \kappa, M \restriction I_i$ is universal.
2) If M is (λ, κ)-i.md.-limit or just weakly (λ, κ)-i.md-limit and $a <_M b$ <u>then</u> for some N:

- (a) $N \subseteq M \restriction (a, b)_M$
- (b) $N \in \mathrm{EC}_{T_{\mathrm{rd}}}(\lambda)$ is universal
- (c) every $(C_1, C_2) \in \mathrm{cut}_\kappa(N)$ is realized in M. (but not used)

3.6 Claim. *If $S \subseteq S^{\lambda^+}_\kappa$ is stationary and $M \in EC_{T_{\mathrm{rd}}}(\lambda)$ is (λ, S)-wk-limit <u>then</u> M is (λ, κ)-i.md-limit.*

Proof. Let $\mathbf{F}_1$ witness that M is (λ, S)-wk-limit. We can find $\bar M = \langle M_\alpha : \alpha < \lambda^+\rangle$ so $M_\alpha \in \mathrm{EC}_{T_{\mathrm{or}}}(\lambda)$ is a $\subseteq$-increasing continuous sequence such that $\bar M$ obeys F, such that in addition the sequence is as in the proof of 1.1. So by the choice of $\mathbf{F}$ the set $S' = \{\delta \in S : M_\delta \cong M\}$ is stationary, and by 1.1 the set $S'' = \{\delta : M_\delta$ is (λ, κ)-i.-md-limit$\}$ is $\equiv S^{\lambda^+}_\kappa \mod \mathscr{D}_{\lambda^+}$. Together $S' \cap S'' \neq \emptyset$ hence M is (λ, κ)-i.md.-limit.

3.7 Definition. 1) We say that $\bar M$ witnesses that M is (λ, κ)-i.md.-limit when:

- $(*)$ $\bar M = \langle M_\alpha : \alpha \le \kappa\rangle$ is such that $\circledast_{\bar M}$ from the proof of 1.1 holds and $M = M_\kappa$.

3.8 Claim. *For $\lambda = \lambda^{<\lambda} > \kappa = \mathrm{cf}(\kappa)$ <u>then</u> there is no (λ, κ)-superlimit model.*

Proof. Assume there is one, then by §1 it is a (λ, κ)-i.md.-limit model so there is $\bar M = \langle M_i : i \le \kappa\rangle$ which witnesses this (i.e. such that $\circledast^\kappa_{\bar M}$ from the proof of 1.1). As M_0 is universal for $\mathrm{EC}_{T_{\mathrm{rd}}}(\lambda)$, we can find $c_\eta \in M_0$ for $\eta \in {}^{\kappa \ge}(\lambda + 1)$ such that

$\eta <_{\rm lex} \nu \Rightarrow c_\eta <_{M_0} c_\nu$. For $\zeta < \kappa$ let $\Lambda_\zeta = \{\eta \in {}^\kappa(\lambda+1)$: for every $\varepsilon \in [\zeta,\kappa)$ we have $\eta(\varepsilon) = \lambda\}$ and let $\Lambda_\kappa = \Lambda = \cup\{\Lambda_\zeta : \zeta < \kappa\}$ so $\langle \Lambda_\zeta : \zeta < \kappa\rangle$ is $\subseteq$-increasing. For $\eta \in \Lambda$ let $(C_{1,\eta}, C_{2,\eta})$ be the cut of M_κ with $C_{1,\eta} = \{a \in M_\kappa : a <_{M_\kappa} c_{\eta \restriction i}$ for some $i < \kappa\}$. So $\mathrm{cf}(C_{1,\eta}, C_{2,\eta}) = (\kappa,\kappa)$ recalling clause $(i)_1$ of $\circledast^\kappa_{\bar M}$ from the proof of 1.1.

Let $\langle d_j : j < \lambda\rangle$ be a decreasing sequence in M_0 and let

$\circledast_0$ $M'_i = M_i \restriction \{d : d_j <_M d$ for some $j < \lambda\}$ for $i \le \kappa$.

We can choose M^* such that:

$\circledast_1$ (a) $M_\kappa \subseteq M_* \in \mathrm{EC}_{T_{\rm rd}}(\lambda)$
(b) if $c \in M_* \backslash M_\kappa$ then for some $\eta \in \Lambda, c$ realizes the cut $(C_{1,\eta}, C_{2,\eta})$
(c) for every $\eta \in \Lambda$ there is an isomorphism f_η from M'_κ onto $M_*^{[(C_{1,\eta},C_{2,\eta})]}$

$\circledast_2$ for $\zeta \le \kappa$ let $M^*_\zeta = M_* \restriction \{c : c \in M_\kappa$ or $c \in M^*$ realizes the cut $(C_{1,\eta}, C_{2,\eta})$ for some $\eta \in \Lambda_\zeta\}$.

So

$\circledast_3$ $\langle M^*_\zeta : \zeta \le \kappa\rangle$ is $\subseteq$-increasing continuous (notice that we didn't demand continuity) and $M^*_\kappa = M^*$.

So it is enough to prove that M^*_ζ is (λ,κ)-i.md.-limit for $\zeta < \kappa$ but not for $\zeta = \kappa$.

$\odot_1$ $M^*_\kappa = M^*$ is not a (λ,κ)-i.-md.-limit model.

Why? Assume toward contradiction that there is an isomorphism g from M_κ onto M^*_κ and let $N_i := g(M_i)$ for $i < \kappa$, and let $h : M^*_\kappa \to \kappa$ be $h(c) = \min\{i < \kappa : c \in N_{i+1}\}$. Fix $\eta \in \Lambda$ for a while and let $(C'_{1,\eta}, C'_{2,\eta})$ be the cut of $M^*_\kappa = M^*$ with $C'_{1,\eta} := \{c \in M^* : c <_{M^*} c_{\eta\restriction\zeta}$ for some $\zeta < \kappa\}$. Clearly $\langle c_{\eta\restriction\zeta} : \zeta < \kappa\rangle$ is an increasing unbounded sequence of members of $C'_{1,\eta}$ and $\langle f_\eta(d_\alpha) : \alpha < \lambda\rangle$ (f_η is from $\circledast_1(c)$) is a decreasing sequence of members of $C'_{2,\eta}$ unbounded from below. So $\mathrm{cf}(C'_{1,\eta}, C'_{2,\eta}) = (\kappa,\lambda)$. This implies that for some $i = i(\eta) < \kappa$, the set $C'_{2,\eta} \cap N_i$ is unbounded from below in $C'_{2,\eta}$. Hence there is an increasing continuous function $h_\eta : \kappa \to \kappa$ such that: $\cup\{(c_{\eta\restriction h_\eta(i)}, c_{\eta\restriction j})_{M^*_\kappa} : j \in [h_\eta(i),\kappa)\}$ is disjoint to N_i. Now we choose (η_ζ, ξ_ζ) by induction on $\zeta < \kappa$ such that:

$\circledast_4$ (a) $\xi_\zeta < \kappa$ and $\eta_\zeta \in \Lambda_{\xi_\zeta}$
(b) if $\zeta_1 < \zeta_2 < \kappa$ then $\eta_{\zeta_1} \restriction \xi_{\zeta_1} \triangleleft \eta_{\zeta_2}$ and $\xi_{\zeta_1} < \xi_{\zeta_2}$
(c) the set $(\cup\{[c_{\eta_\zeta\restriction\xi_\zeta}, c_{\eta_\zeta\restriction\xi})_{M^*_\kappa} : \xi \in (\xi_{\zeta+1},\kappa)\})$ is disjoint to N_ζ
(d) if ζ is a successor then ξ_ζ is a successor
(e) if $\eta_{\zeta+1}\restriction\xi_{\zeta+1} \triangleleft \nu \in {}^{\kappa\ge}(\lambda+1)$ then $c_\nu \in (C_{\eta_{\zeta+1}\restriction(\xi_{\zeta+1}-1)\hat{}<1>}, C_{\eta_{\zeta+1}\restriction(\xi_{\zeta+1}-1)\hat{}<2>})_{M_\kappa}$ is disjoint to N_ζ.

There is no problem to carry the induction:

Case 1: $\zeta = 0$.
Choose $\xi_\zeta = 0, \eta_\zeta \in \Lambda_0$.

Case 2: $\zeta = \zeta_1 + 1$.
Choose $\xi_\zeta = h_{\eta_{\zeta_1}}(\xi_{\zeta_1}) + 6$.
Choose η_ζ such that

$$\eta_\zeta \restriction (h_{\eta_{\zeta_1 (\xi_{\zeta_1})+5)}\hat{}<1>} \trianglelefteq \eta_\zeta \in \Lambda_{\xi_\zeta}.$$

Case 3: ζ limit.
$\xi_\zeta = \cup\{\xi_\alpha : \alpha < \zeta\}$.
Choose $\eta_\zeta \in \Lambda_{\xi_\zeta + 1}$ such that $\alpha < \zeta \Rightarrow \eta_\alpha \restriction \xi_\alpha \trianglelefteq \eta_\zeta$.
Let $\eta = \cup\{\eta_\zeta \restriction \xi_\zeta : \zeta < \kappa\}$. So $\eta \in {}^\kappa(\lambda + 1)$ and $c_\eta \notin N_\zeta$ for every $\zeta < \kappa$ but $\cup\{N_\zeta : \zeta < \kappa\} = M^*_\kappa = M^*$, contradiction.

$\boxdot$ M^*_ζ is a (λ, κ)-.i.md.-limit model for $\zeta < \kappa$.

Why? We define $M_{\zeta,i} \subseteq M^*_\zeta$ for $i < \kappa$ by: $c \in M_{\zeta,i}$ iff one of the following occurs

- (a) $c \in M_i$ but for no $\eta \in \Lambda_\zeta$ do we have $c \in B_\eta := \cup\{[c_{\eta\restriction(\zeta+i)}, c_{\eta\restriction\varepsilon})_{M_i} : \varepsilon \in (\zeta + i, \kappa)\}$
- (b) $c \in f_\eta(M'_i)$ for some $\eta \in \Lambda_\zeta$.

Let

$$J_{\zeta,\eta} = \cup\{(C_{\eta\restriction\zeta}, C_{\eta\restriction\varepsilon})_{M^*_\kappa} : \varepsilon \in (\zeta, \kappa)\}$$

$$J_{\zeta,\eta,\varepsilon} = (C_{\eta\restriction\zeta}, C_{\eta\restriction\varepsilon})$$

$$\langle J_{\zeta,\eta} : \eta \in \Lambda_\zeta \rangle \text{ are pairwise disjoint}$$

$$J_{\zeta,\eta,\varepsilon} \text{ is an initial segment of } J_{\zeta,\eta}$$

$$J_{\zeta,\eta} = \cup\{J_{\zeta,\eta,\varepsilon} : \varepsilon \in (\zeta, \kappa)\}.$$

We will make $M_{\zeta,i} \cap J_{\zeta,\eta}$ bounded in $J_{\zeta,\eta}$ for each $i < \kappa$.
Now $\langle M_{\zeta,i} : i < \kappa \rangle$ is a (λ, κ)-sequence, see Definition 3.3 hence by 3.4(1) the model M^*_ζ is a (λ, κ)-i.md.-limit model.

3.9 Theorem. *If $\lambda = \lambda^{<\lambda} > \kappa = \mathrm{cf}(\kappa)$ then T_{rd} has a (λ,κ)-limit$^+$ model, i.e.: if $\langle M_\alpha : \alpha < \lambda^+\rangle$ is $\subseteq$-increasing continuous sequence of models $\in \mathrm{EC}_{T_{\mathrm{rd}}}(\lambda)$ and $M_{\alpha+1}$ is (λ,κ)-i.md.-limit model for every $\alpha < \lambda^+$ then for a club of $\delta < \lambda^+$ if $\mathrm{cf}(\delta) = \kappa$ then M_δ is a (λ,κ)-i.md.-limit.*

Proof. Let $\langle M_\alpha : \alpha < \lambda^+\rangle$ be as in the theorem and $M = \bigcup\limits_{\alpha<\lambda^+} M_\alpha$, without loss of generality $\|M\| = \lambda^+$. As $\lambda = \lambda^{<\lambda}$ by 3.4(4) we can find a club E of λ^+ such that:

- $\circledast$ if $\alpha < \delta \in E$ and (C_1,C_2) is a cut of M_α of cofinality $\neq (\lambda,\lambda)$ and some $a \in M$ realizes the cut then some $a \in M_\delta$ realizes the cut.

Let $\langle \alpha_\varepsilon : \varepsilon \le \kappa\rangle$ be an increasing continuous sequence of ordinals from E and we shall prove that M_{α_κ} is (λ,κ)-i.md.-limit; this suffices (really just $\alpha_\kappa \in E$ suffice).

Now $M_{\alpha_\kappa+1}$ is (λ,κ)-i.md.-limit hence there is an $\subseteq$-increasing sequence $\langle M_{\alpha_\kappa+1,i} : i < \kappa\rangle$ witnessing $M_{\alpha_\kappa+1}$ is (λ,κ)-i.md.-limit model. Now $M_{\alpha_\kappa+1,i} \cap M_{\alpha_\kappa} = \cup\{M_{\alpha_{\kappa+1},i} \cap M_{\alpha_\zeta} : \zeta < \kappa\}$ hence without loss of generality $M_{\alpha_\kappa,0} \cap M_{\alpha_0}$ has cardinality λ hence $N_i := M_{\alpha_\kappa+1,i} \cap M_{\alpha_i} \in \mathrm{EC}_{T_{\mathrm{rd}}}(\lambda)$.

Clearly

- $(*)_1$ $\langle N_i : i < \kappa\rangle$ is a $\subseteq$-increasing sequence of members of $\mathrm{EC}_{T_{\mathrm{rd}}}(\lambda)$ with union M_{α_κ}.

So it is enough to show that $\langle N_i : i < \kappa\rangle$ is a (λ,κ)-sequence by 3.4(1). By $(*)_1$, clause (a) from Definition 3.3 holds.

- $\circledast_2$ $\langle N_i : i < \kappa\rangle$ satisfies clause (b) of 3.3.

[Why? Let $i < \kappa$ and (C_1,C_2) be a cut of N_i of cofinality $\neq (\lambda,\lambda)$. As $C_1,C_2 \subseteq N_i \subseteq M_{\alpha_{\kappa+1},i}$ by the properties of $\langle M_{\alpha_\kappa+1,i} : i < \kappa\rangle$ there is $a \in M_{\alpha_\kappa+1}$ such that $C_1 < a < C_2$. If $\exists b \in M_{\alpha_i}, C_1 < b < C_2$ then $a \in M_{\alpha_\kappa}$ and we are done. If not, a induces on M_{α_i} a cut $(C_1',C_2'), C_1 \subseteq C_1', C_2 \subseteq C_2'$, $\mathrm{cf}(C_1',C_2') \neq \mathrm{cf}(C_1,C_2) \neq \lambda$. As $\alpha_i < \alpha_\kappa \in E$, by $\circledast$ there is $a \in M_{\alpha_\kappa}$ such that $C_1 < a < C_2$. So clause (b) of Definition 3.3 really holds.]

- $\circledast_3$ if $a <_{M_{\alpha_\kappa}} b$ then $M_{\alpha_\kappa} \restriction (a,b)$ is universal (for $(\mathrm{EC}_{T_{\mathrm{or}}}(\lambda),\subseteq)$).

[Why? As $\langle \alpha_\varepsilon : \varepsilon \le \kappa\rangle$ is increasing continuous and $\langle M_\alpha : \alpha < \lambda^+\rangle$ is increasing continuous, clearly for some $i < \kappa$ we have $a,b \in M_{\alpha_i}$ hence $M_{\alpha_i+1} \restriction (a,b)$ is λ-universal but $M_{\alpha_i+1} \subseteq M_{\alpha_k}$ so $M_{\alpha_\kappa} \restriction (a,b)$ is universal so we are done.]

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